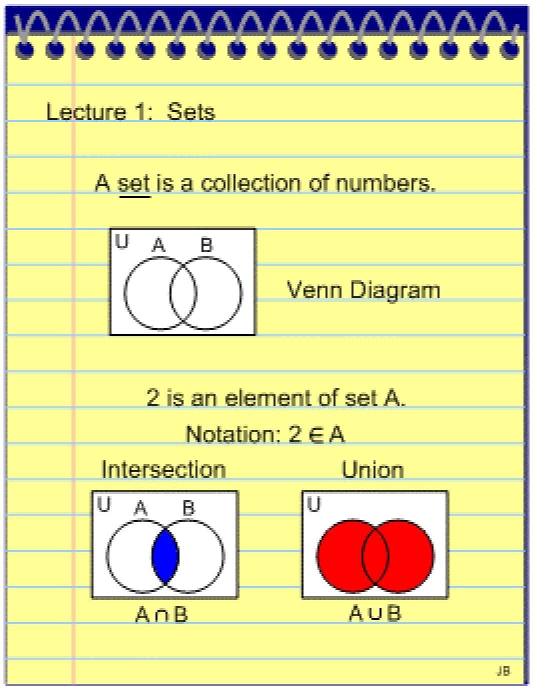
COLLEGE ENTRANCE EXAM MATH PREPARATION NOTES

Lecture 1 Notes

CEMP001-01



Lecture 1: Sets

A set is a collection of numbers.

Venn Diagram

2 is an element of set A.

Notation: $2 \in A$

Intersection

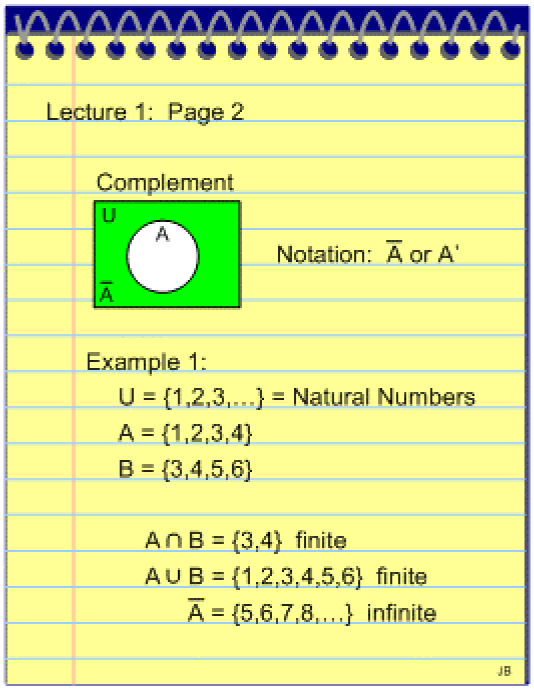
Union

$A \cap B$

$A \cup B$

JB

CEMP001-02



Lecture 1: Page 2

Complement

Notation: $\overline{A}$ or A'

Example 1:

U = {1,2,3,…} = Natural Numbers

A = {1,2,3,4}

B = {3,4,5,6}

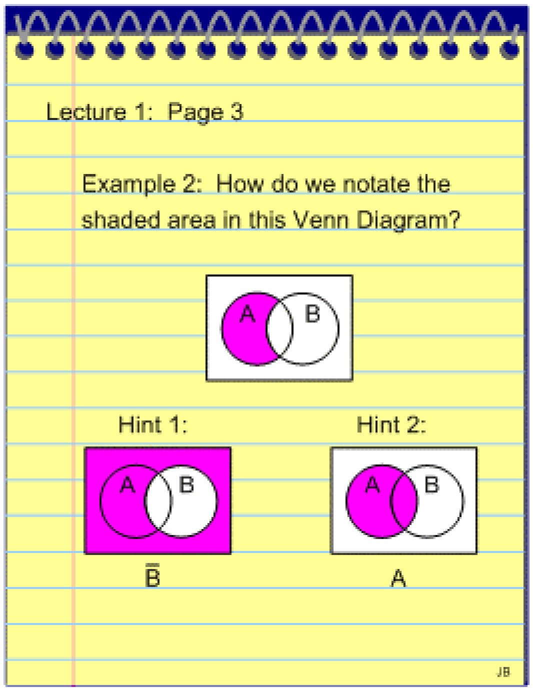
$A \cap B = \{3,4\}$ finite

$A \cup B = \{1,2,3,4,5,6\}$ finite

$\overline{A} = \{5,6,7,8,\ldots\}$ infinite

JB

CEMP001-03



Lecture 1: Page 3

Example 2: How do we notate the shaded area in this Venn Diagram?

Hint 1:

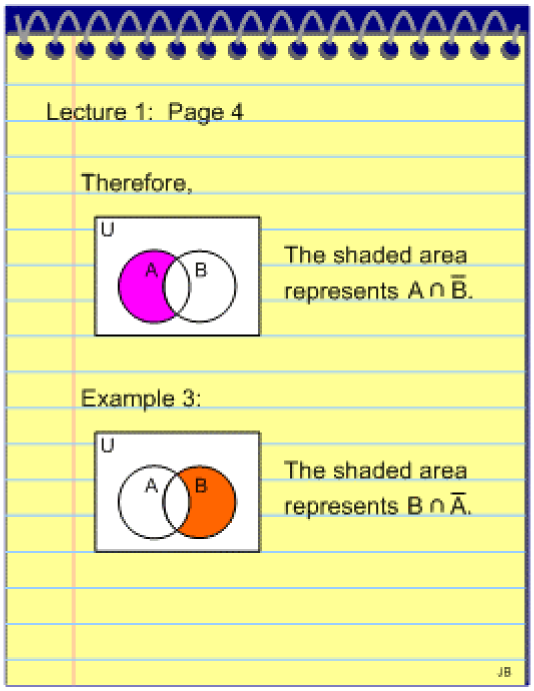
Hint 2:

$\overline{B}$

A

JB

CEMP001-04



Lecture 1: Page 4

Therefore,

The shaded area represents $A \cap \overline{B}$.

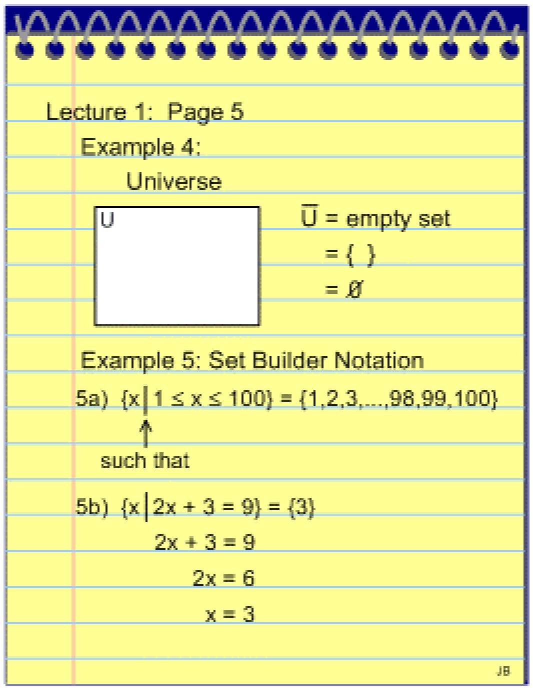
Example 3:

The shaded area represents $B \cap \overline{A}$.

JB

Lecture 1 Notes, Continued

CEMP001-05



Lecture 1: Page 5

Example 4:

Universe

$\overline{U}$ = empty set

= { }

= $\emptyset$

Example 5: Set Builder Notation

5a) $\{x \mid 1 \le x \le 100\} = \{1,2,3,\ldots,98,99,100\}$

↑ such that

5b) $\{x \mid 2x + 3 = 9\} = \{3\}$

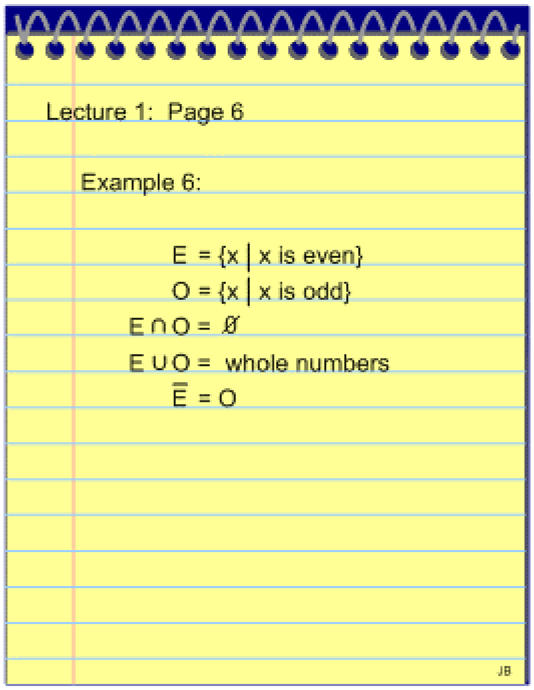
$2x + 3 = 9$

$2x = 6$

$x = 3$

JB

CEMP001-06



Lecture 1: Page 6

Example 6:

$E = \{x \mid x \text{ is even}\}$

$O = \{x \mid x \text{ is odd}\}$

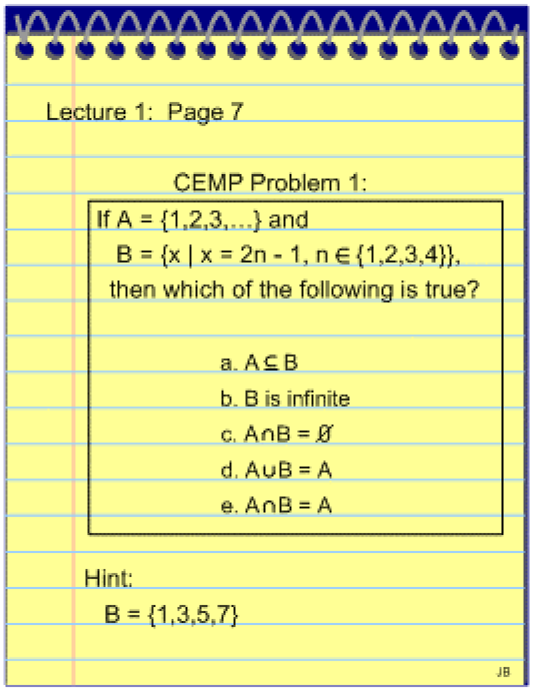
$E \cap O = \emptyset$

$E \cup O =$ whole numbers

$\overline{E} = O$

JB

CEMP001-07



Lecture 1: Page 7

CEMP Problem 1:

If $A = \{1,2,3,\ldots\}$ and $B = \{x \mid x = 2n - 1, n \in \{1,2,3,4\}\}$, then which of the following is true?

a. $A \subseteq B$

b. B is infinite

c. $A \cap B = \emptyset$

d. $A \cup B = A$

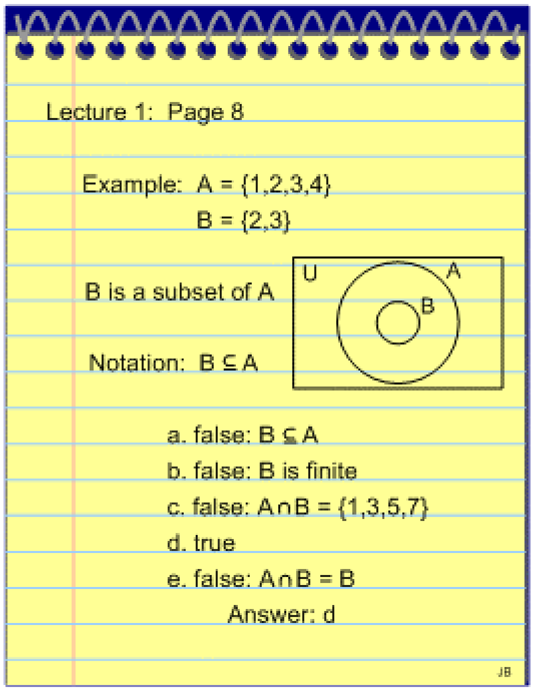
e. $A \cap B = A$

Hint:

$B = \{1,3,5,7\}$

JB

CEMP001-08



Lecture 1: Page 8

Example: $A = \{1,2,3,4\}$

$B = \{2,3\}$

B is a subset of A

Notation: $B \subseteq A$

a. false: $B \subseteq A$

b. false: B is finite

c. false: $A \cap B = \{1,3,5,7\}$

d. true

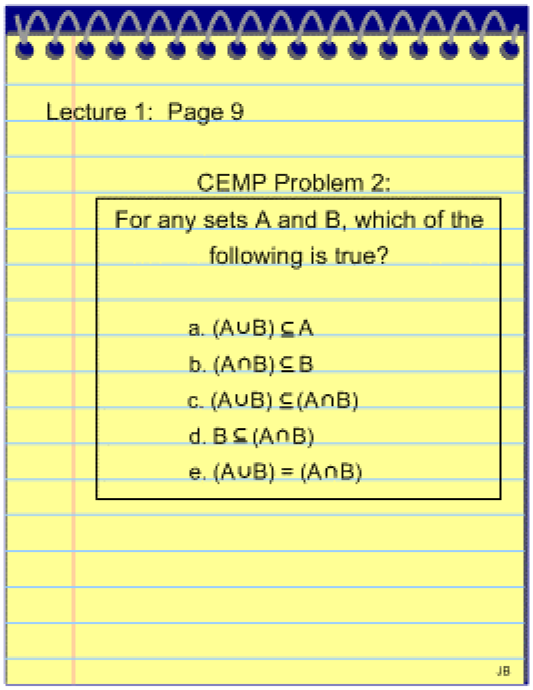
e. false: $A \cap B = B$

Answer: d

JB

Lecture 1 Notes, Continued

CEMP001-09



Lecture 1: Page 9

CEMP Problem 2:

For any sets A and B, which of the following is true?

a. $(A \cup B) \subseteq A$

b. $(A \cap B) \subseteq B$

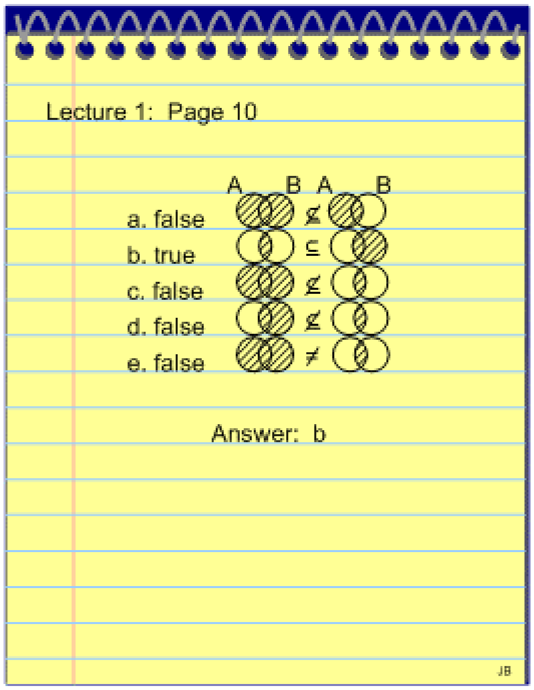
c. $(A \cup B) \subseteq (A \cap B)$

d. $B \subseteq (A \cap B)$

e. $(A \cup B) = (A \cap B)$

JB

CEMP001-10



Lecture 1: Page 10

a. false $\not\subseteq$

b. true $\subseteq$

c. false $\not\subseteq$

d. false $\not\subseteq$

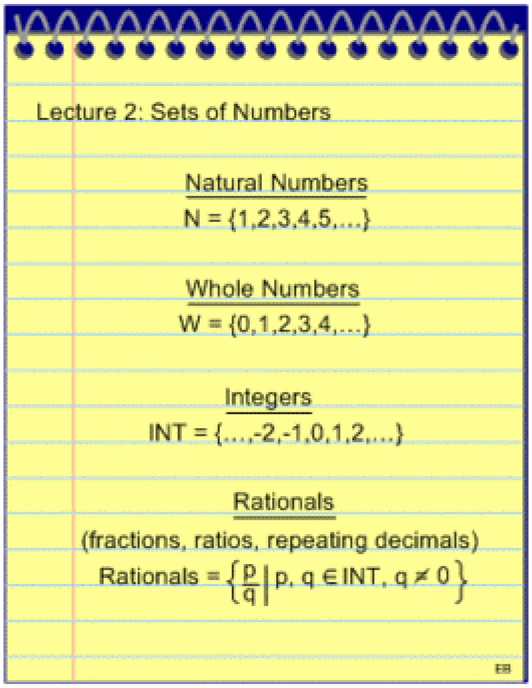
e. false $\neq$

Answer: b

JB

Lecture 2 Notes

CEMP002-01



Lecture 2: Sets of Numbers

Natural Numbers

N = {1,2,3,4,5,...}

Whole Numbers

W = {0,1,2,3,4,...}

Integers

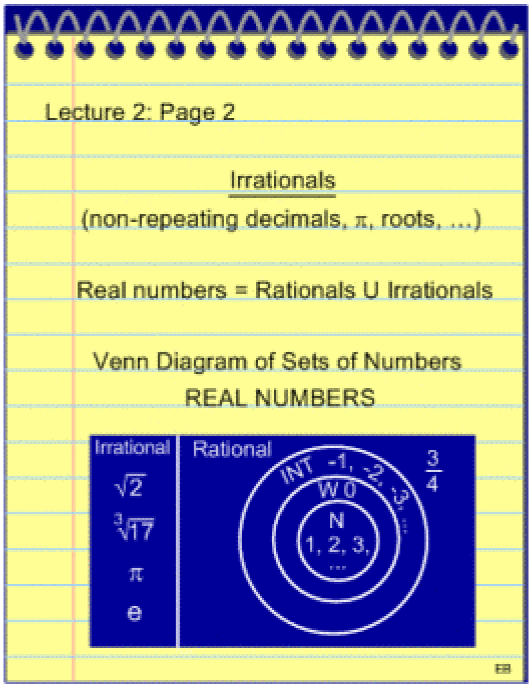
INT = {...,-2,-1,0,1,2,...}

Rationals

(fractions, ratios, repeating decimals)

$$\text{Rationals} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \text{INT}, q \neq 0 \right\}$$

CEMP002-02



Lecture 2: Page 2

Irrationals

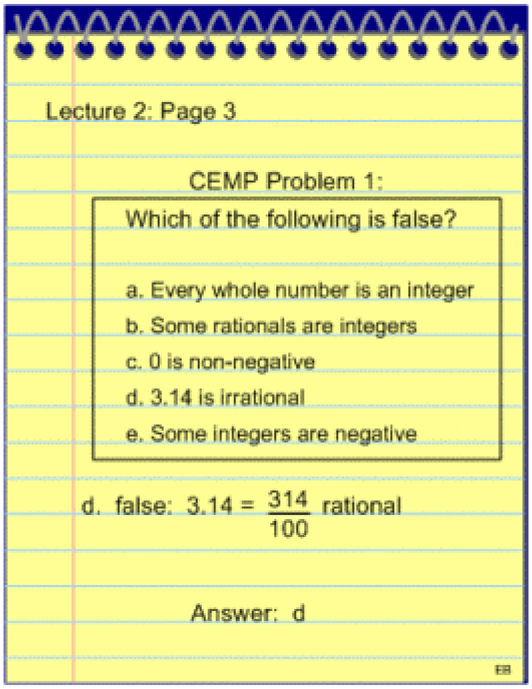
(non-repeating decimals, π, roots, ...)

Real numbers = Rationals U Irrationals

Venn Diagram of Sets of Numbers

REAL NUMBERS

CEMP002-03



Lecture 2: Page 3

CEMP Problem 1:

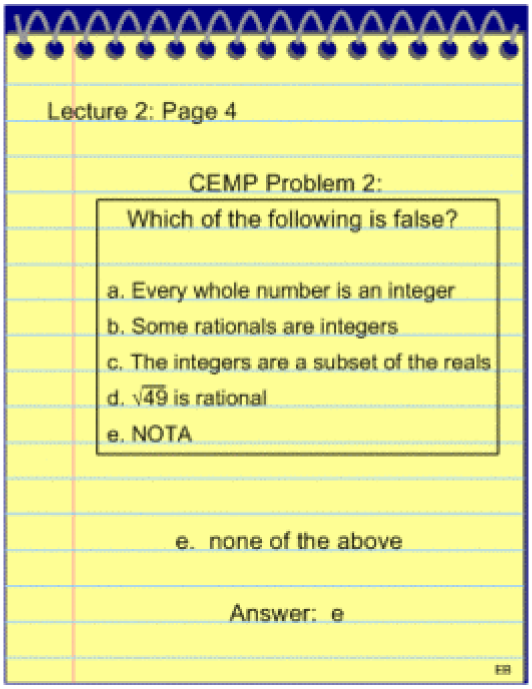
Which of the following is false?

a. Every whole number is an integer
b. Some rationals are integers
c. 0 is non-negative
d. 3.14 is irrational
e. Some integers are negative

d. false: $3.14 = \frac{314}{100}$ rational

Answer: d

CEMP002-04



Lecture 2: Page 4

CEMP Problem 2:

Which of the following is false?

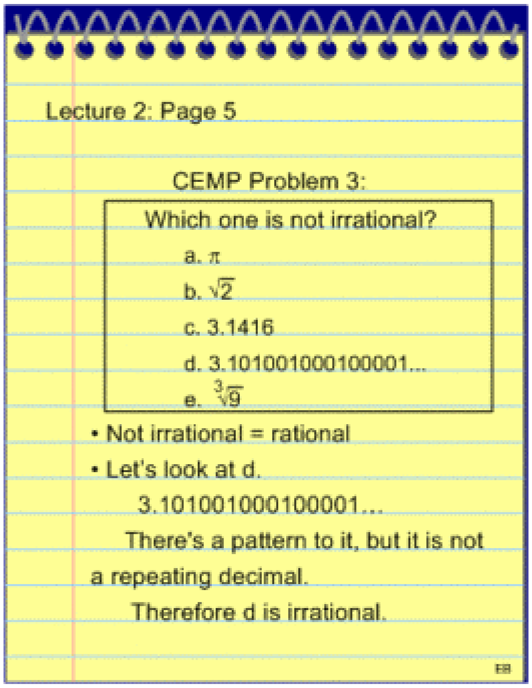
a. Every whole number is an integer
b. Some rationals are integers
c. The integers are a subset of the reals
d. $\sqrt{49}$ is rational
e. NOTA

e. none of the above

Answer: e

Lecture 2 Notes, Continued

CEMP002-05



Lecture 2: Page 5

CEMP Problem 3:

Which one is not irrational?

a. π

b. $\sqrt{2}$

c. 3.1416

d. 3.101001000100001...

e. $\sqrt[3]{9}$

- Not irrational = rational
- Let's look at d.

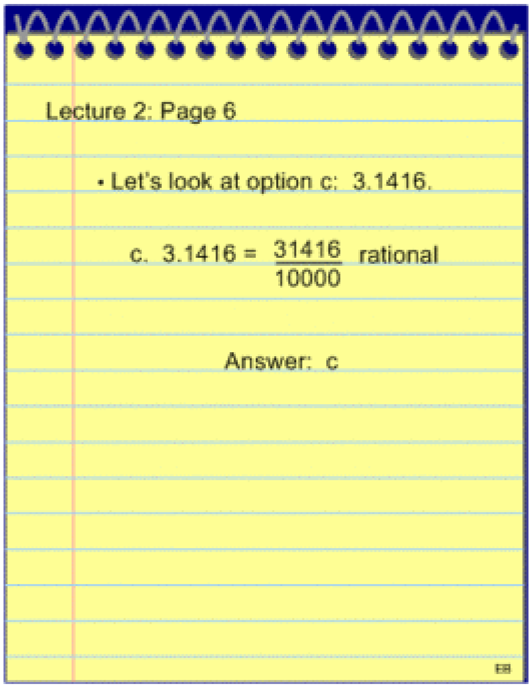
3.101001000100001...

There's a pattern to it, but it is not a repeating decimal.

Therefore d is irrational.

EB

CEMP002-06



Lecture 2: Page 6

- Let's look at option c: 3.1416.

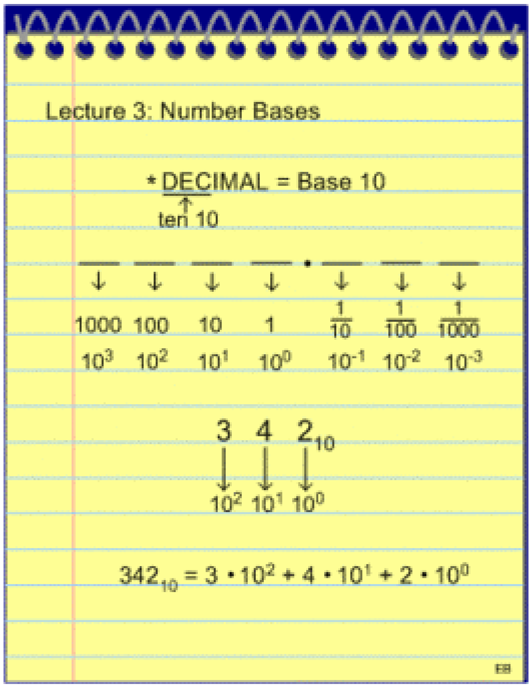
c. $3.1416 = \frac{31416}{10000}$ rational

Answer: c

EB

Lecture 3 Notes

CEMP003-01



Lecture 3: Number Bases

* DECIMAL = Base 10

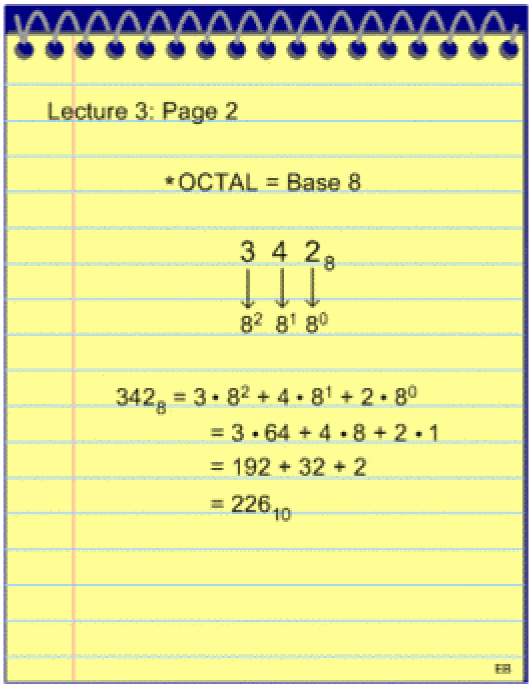
$\overset{\uparrow}{\text{ten}}$ 10

___	___	___	___	.	___	___	___
↓	↓	↓	↓		↓	↓	↓
1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}

$$\begin{tikzcd}3 \arrow[d] & 4 \arrow[d] & 2_{10} \arrow[d] \\ 10^2 & 10^1 & 10^0\end{tikzcd}$$

$342_{10} = 3 \cdot 10^2 + 4 \cdot 10^1 + 2 \cdot 10^0$

CEMP003-02



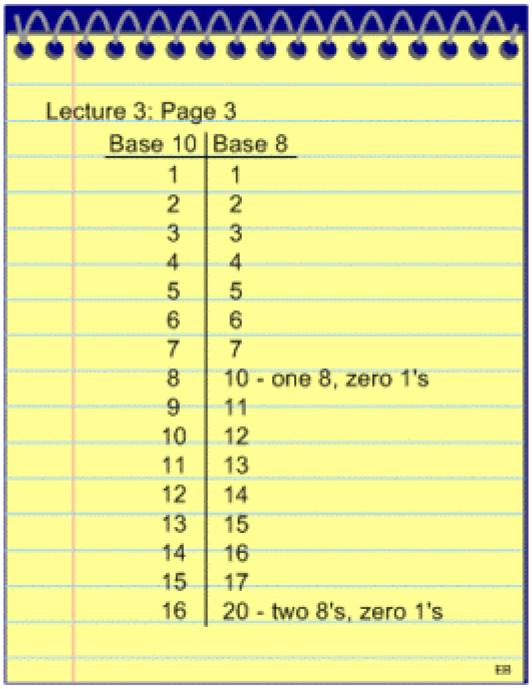
Lecture 3: Page 2

* OCTAL = Base 8

$$\begin{tikzcd}3 \arrow[d] & 4 \arrow[d] & 2_8 \arrow[d] \\ 8^2 & 8^1 & 8^0\end{tikzcd}$$

$$\begin{aligned}342_8 &= 3 \cdot 8^2 + 4 \cdot 8^1 + 2 \cdot 8^0 \\ &= 3 \cdot 64 + 4 \cdot 8 + 2 \cdot 1 \\ &= 192 + 32 + 2 \\ &= 226_{10}\end{aligned}$$

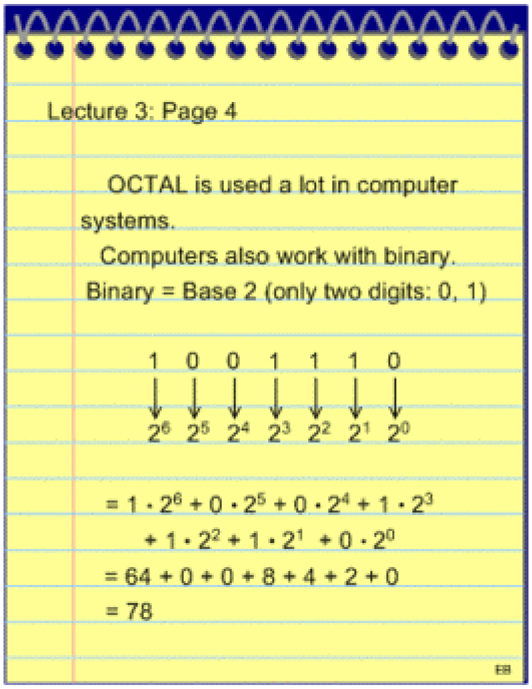
CEMP003-03



Lecture 3: Page 3

Base 10	Base 8
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	10 - one 8, zero 1's
9	11
10	12
11	13
12	14
13	15
14	16
15	17
16	20 - two 8's, zero 1's

CEMP003-04



Lecture 3: Page 4

OCTAL is used a lot in computer systems.

Computers also work with binary.

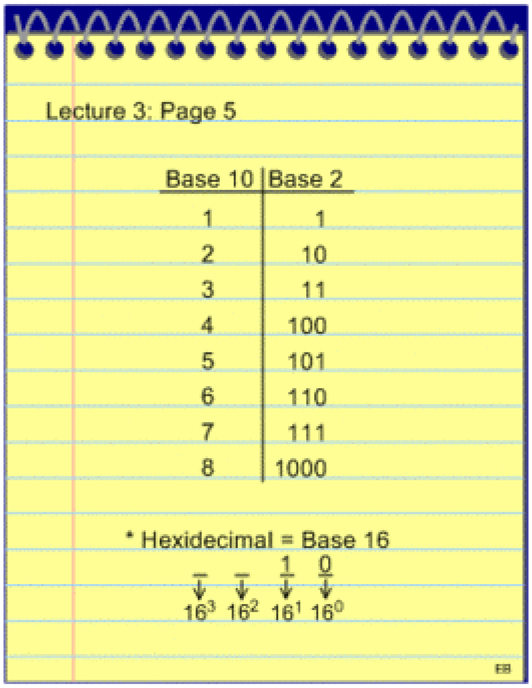
Binary = Base 2 (only two digits: 0, 1)

$$\begin{tikzcd}1 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d] \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0\end{tikzcd}$$

$$\begin{aligned}&= 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 \\ &\quad + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 64 + 0 + 0 + 8 + 4 + 2 + 0 \\ &= 78\end{aligned}$$

Lecture 3 Notes, Continued

CEMP003-05



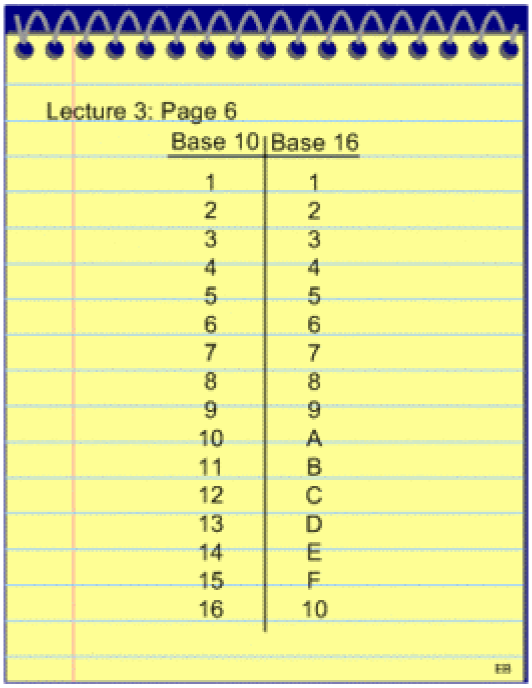
Lecture 3: Page 5

Base 10	Base 2
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000

* Hexidecimal = Base 16

$$\underset{16^3}{\underset{\downarrow}{\underline{\ \ }}} \quad \underset{16^2}{\underset{\downarrow}{\underline{\ \ }}} \quad \underset{16^1}{\underset{\downarrow}{\underline{1}}} \quad \underset{16^0}{\underset{\downarrow}{\underline{0}}}$$

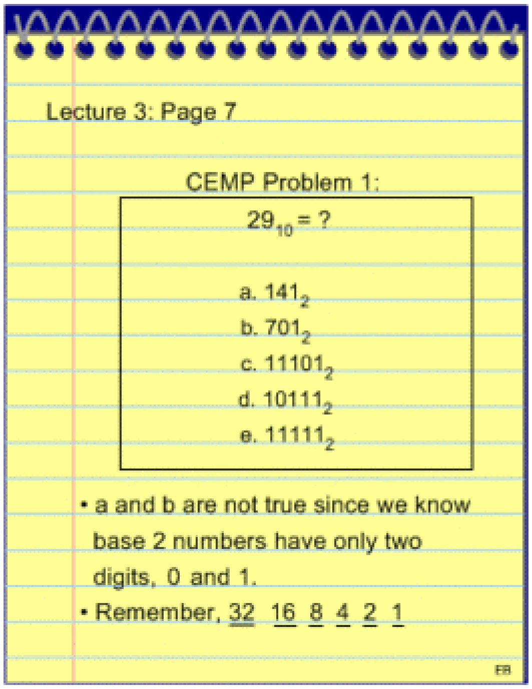
CEMP003-06



Lecture 3: Page 6

Base 10	Base 16
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	A
11	B
12	C
13	D
14	E
15	F
16	10

CEMP003-07



Lecture 3: Page 7

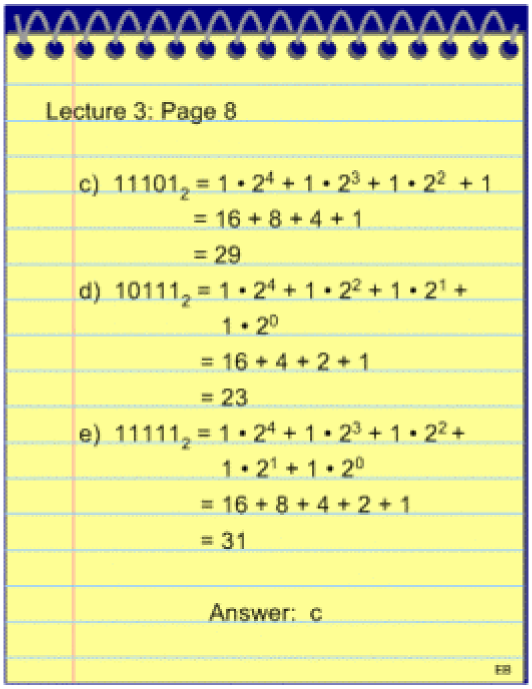
CEMP Problem 1:

$29_{10} = ?$

a. 141_2
b. 701_2
c. 11101_2
d. 10111_2
e. 11111_2

- a and b are not true since we know base 2 numbers have only two digits, 0 and 1.
- Remember, $\underline{32}\ \ \underline{16}\ \ \underline{8}\ \ \underline{4}\ \ \underline{2}\ \ \underline{1}$

CEMP003-08



Lecture 3: Page 8

c) $11101_2 = 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1$
$= 16 + 8 + 4 + 1$
$= 29$

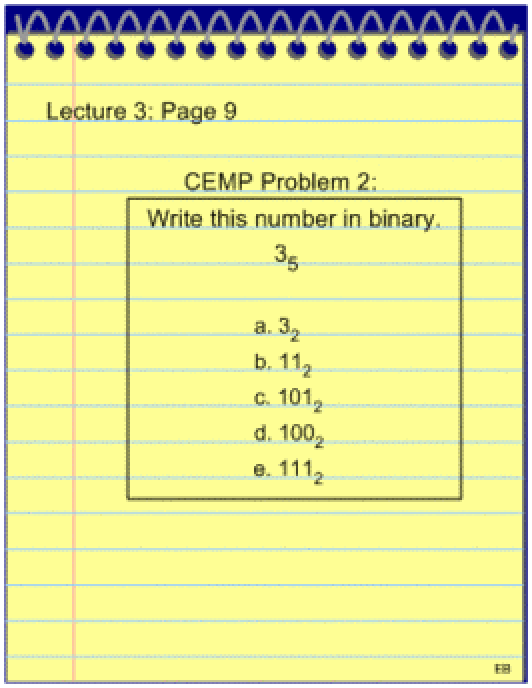
d) $10111_2 = 1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
$= 16 + 4 + 2 + 1$
$= 23$

e) $11111_2 = 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
$= 16 + 8 + 4 + 2 + 1$
$= 31$

Answer: c

Lecture 3 Notes, Continued

CEMP003-09



Lecture 3: Page 9

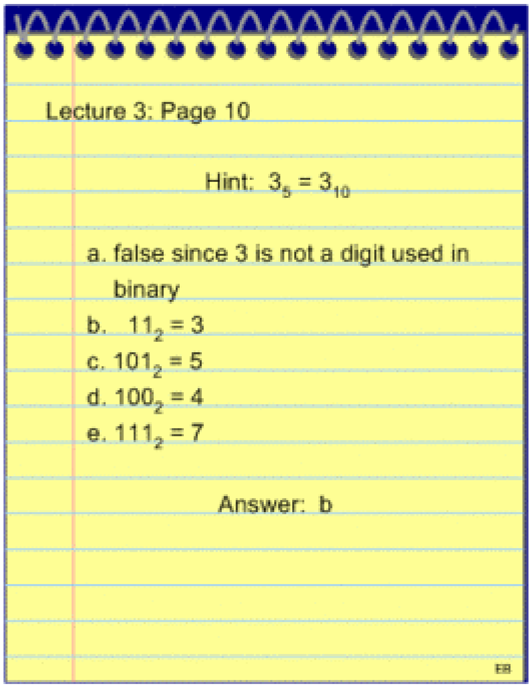
CEMP Problem 2:

Write this number in binary.

3_5

a. 3_2
b. 11_2
c. 101_2
d. 100_2
e. 111_2

CEMP003-10



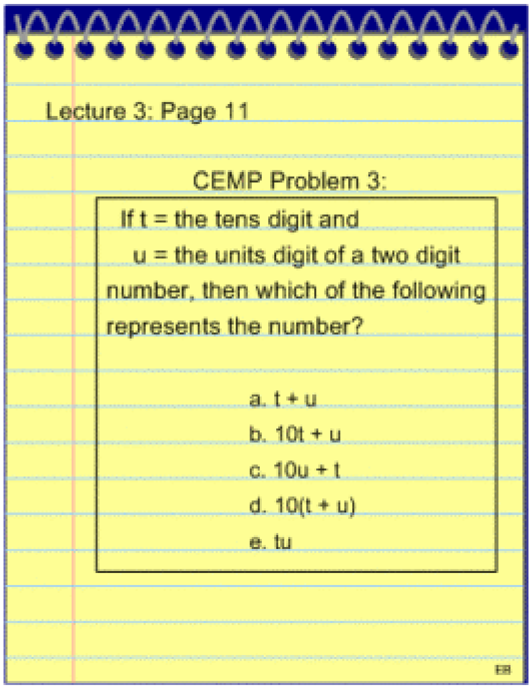
Lecture 3: Page 10

Hint: $3_5 = 3_{10}$

a. false since 3 is not a digit used in binary
b. $11_2 = 3$
c. $101_2 = 5$
d. $100_2 = 4$
e. $111_2 = 7$

Answer: b

CEMP003-11



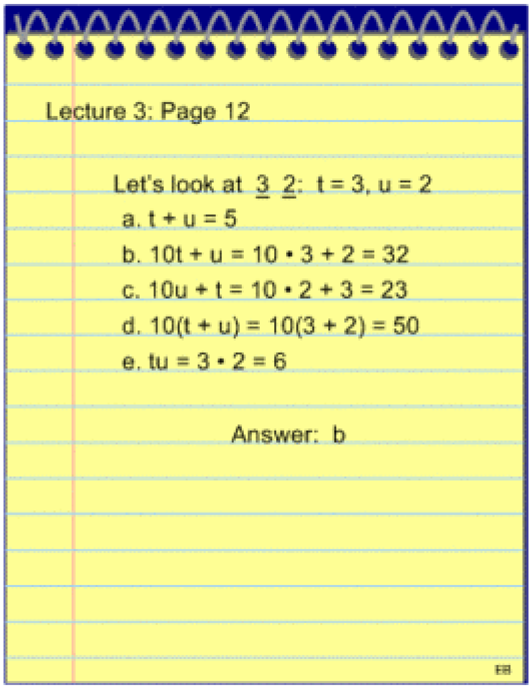
Lecture 3: Page 11

CEMP Problem 3:

If t = the tens digit and u = the units digit of a two digit number, then which of the following represents the number?

a. $t + u$
b. $10t + u$
c. $10u + t$
d. $10(t + u)$
e. tu

CEMP003-12



Lecture 3: Page 12

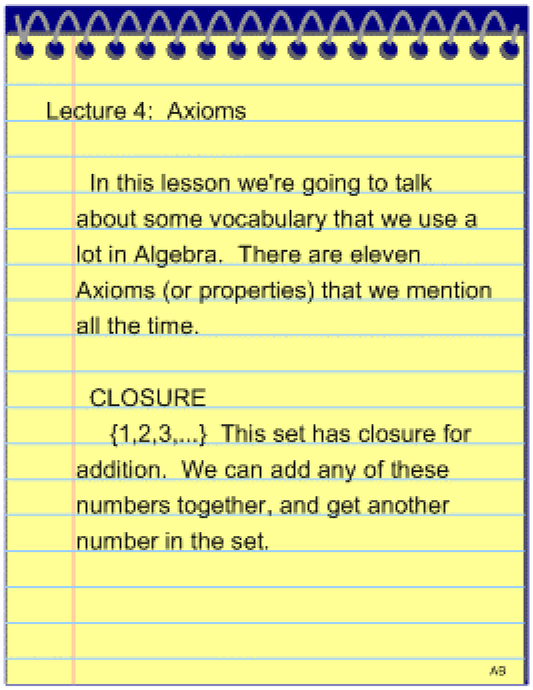
Let's look at $\underline{3}\ \underline{2}$: $t = 3, u = 2$

a. $t + u = 5$
b. $10t + u = 10 \cdot 3 + 2 = 32$
c. $10u + t = 10 \cdot 2 + 3 = 23$
d. $10(t + u) = 10(3 + 2) = 50$
e. $tu = 3 \cdot 2 = 6$

Answer: b

Lecture 4 Notes

CEMP004-01



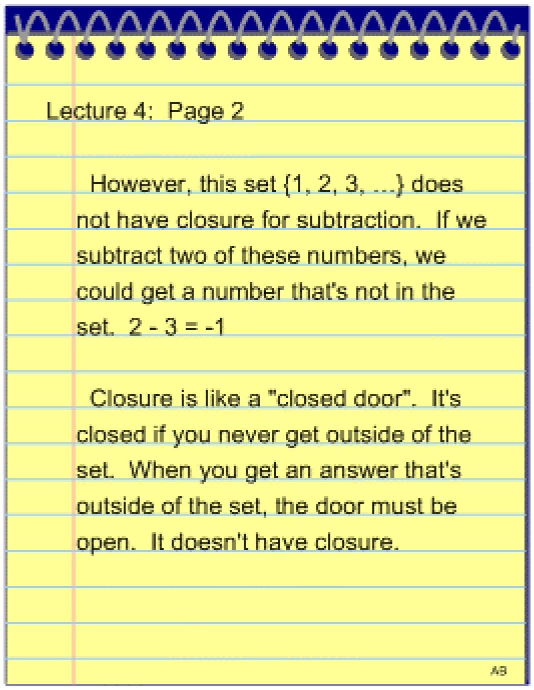
Lecture 4: Axioms

In this lesson we're going to talk about some vocabulary that we use a lot in Algebra. There are eleven Axioms (or properties) that we mention all the time.

CLOSURE

{1,2,3,...} This set has closure for addition. We can add any of these numbers together, and get another number in the set.

CEMP004-02

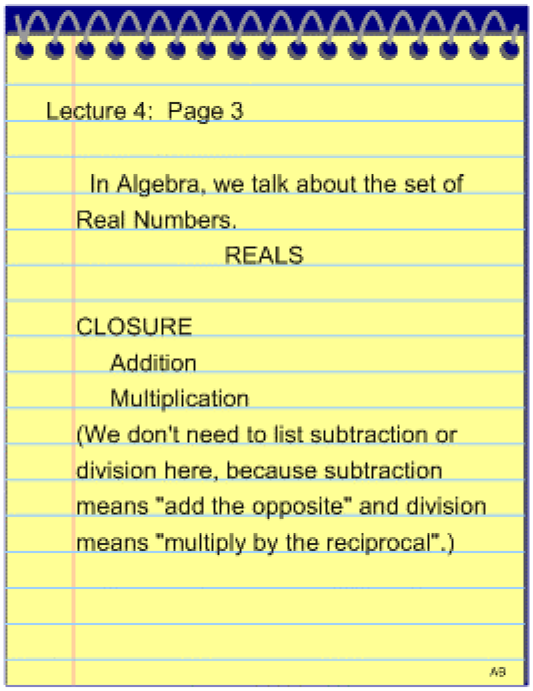


Lecture 4: Page 2

However, this set {1, 2, 3, ...} does not have closure for subtraction. If we subtract two of these numbers, we could get a number that's not in the set. 2 - 3 = -1

Closure is like a "closed door". It's closed if you never get outside of the set. When you get an answer that's outside of the set, the door must be open. It doesn't have closure.

CEMP004-03



Lecture 4: Page 3

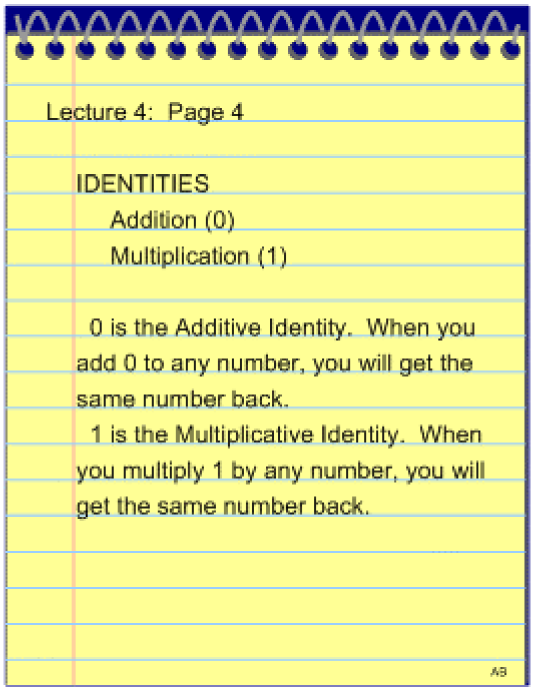
In Algebra, we talk about the set of Real Numbers.

REALS

CLOSURE
Addition
Multiplication

(We don't need to list subtraction or division here, because subtraction means "add the opposite" and division means "multiply by the reciprocal".)

CEMP004-04



Lecture 4: Page 4

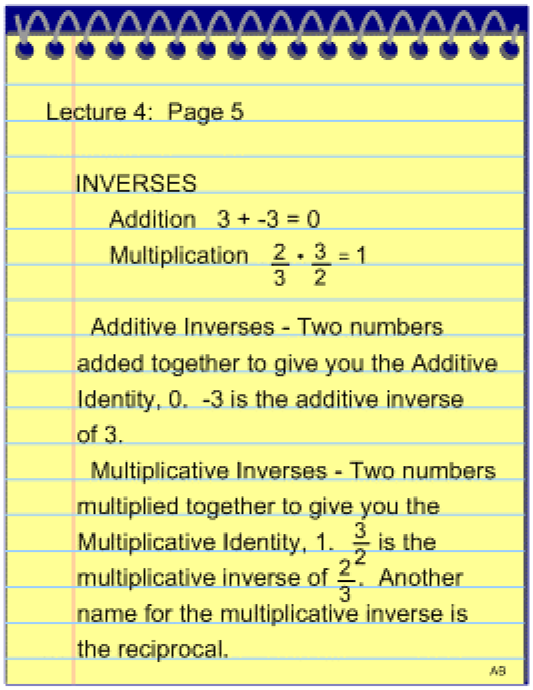
IDENTITIES
Addition (0)
Multiplication (1)

0 is the Additive Identity. When you add 0 to any number, you will get the same number back.

1 is the Multiplicative Identity. When you multiply 1 by any number, you will get the same number back.

Lecture 4 Notes, Continued

CEMP004-05



Lecture 4: Page 5

INVERSES

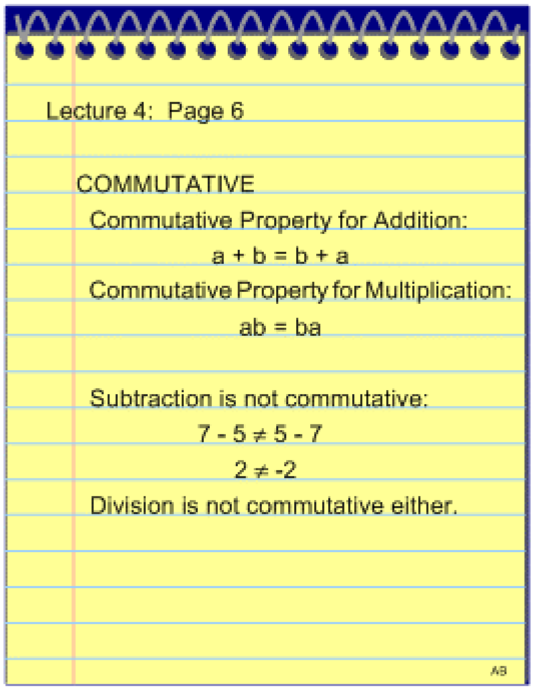
Addition 3 + -3 = 0

Multiplication $\frac{2}{3} \cdot \frac{3}{2} = 1$

Additive Inverses - Two numbers added together to give you the Additive Identity, 0. -3 is the additive inverse of 3.

Multiplicative Inverses - Two numbers multiplied together to give you the Multiplicative Identity, 1. $\frac{3}{2}$ is the multiplicative inverse of $\frac{2}{3}$. Another name for the multiplicative inverse is the reciprocal.

CEMP004-06



Lecture 4: Page 6

COMMUTATIVE

Commutative Property for Addition:

a + b = b + a

Commutative Property for Multiplication:

ab = ba

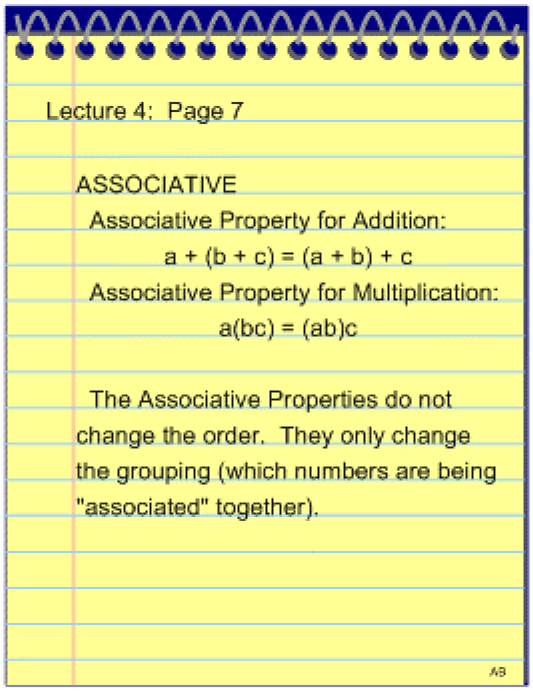
Subtraction is not commutative:

$7 - 5 \neq 5 - 7$

$2 \neq -2$

Division is not commutative either.

CEMP004-07



Lecture 4: Page 7

ASSOCIATIVE

Associative Property for Addition:

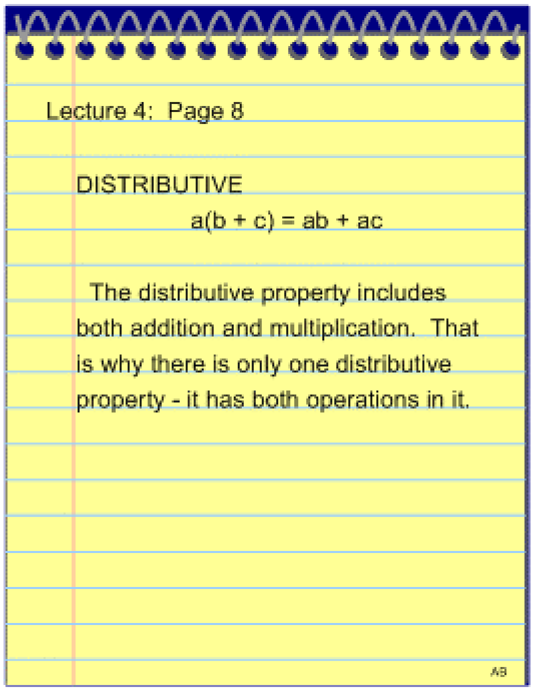
a + (b + c) = (a + b) + c

Associative Property for Multiplication:

a(bc) = (ab)c

The Associative Properties do not change the order. They only change the grouping (which numbers are being "associated" together).

CEMP004-08



Lecture 4: Page 8

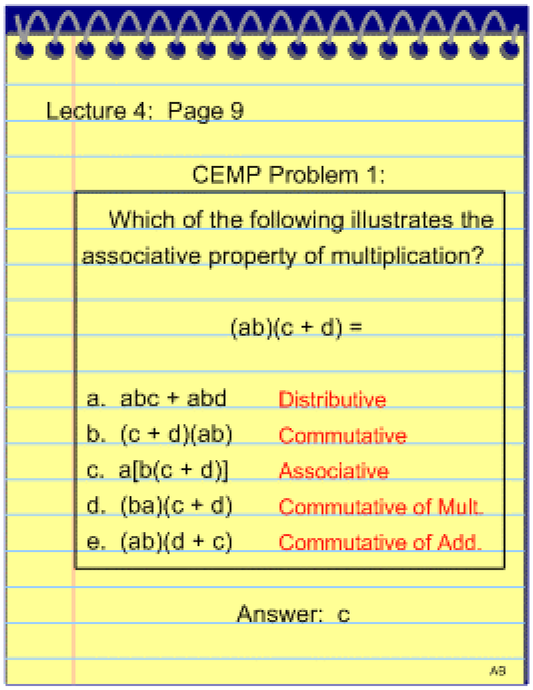
DISTRIBUTIVE

a(b + c) = ab + ac

The distributive property includes both addition and multiplication. That is why there is only one distributive property - it has both operations in it.

Lecture 4 Notes, Continued

CEMP004-09



Lecture 4: Page 9

CEMP Problem 1:

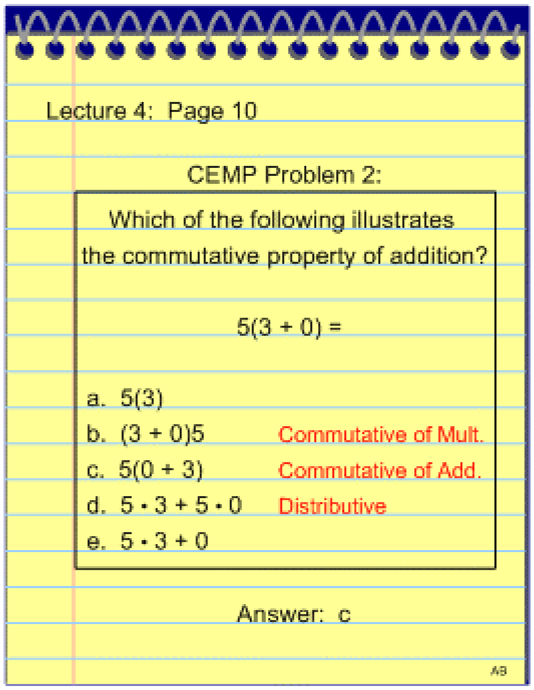
Which of the following illustrates the associative property of multiplication?

$(ab)(c + d) =$

a. $abc + abd$ Distributive
b. $(c + d)(ab)$ Commutative
c. $a[b(c + d)]$ Associative
d. $(ba)(c + d)$ Commutative of Mult.
e. $(ab)(d + c)$ Commutative of Add.

Answer: c

CEMP004-10



Lecture 4: Page 10

CEMP Problem 2:

Which of the following illustrates the commutative property of addition?

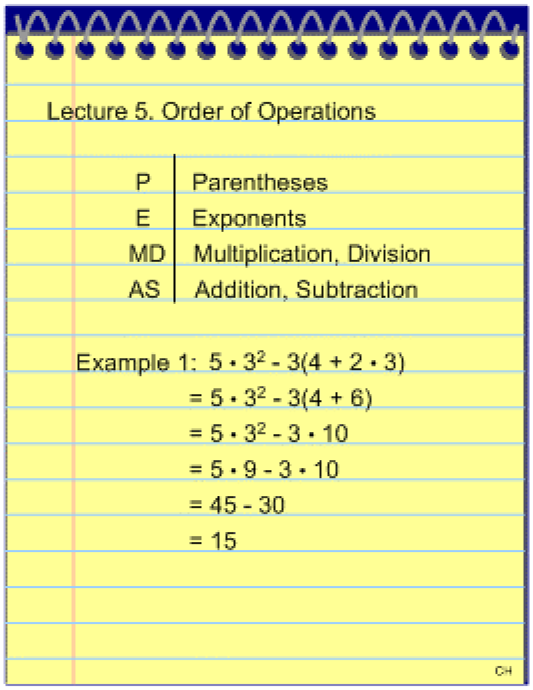
$5(3 + 0) =$

a. $5(3)$
b. $(3 + 0)5$ Commutative of Mult.
c. $5(0 + 3)$ Commutative of Add.
d. $5 \cdot 3 + 5 \cdot 0$ Distributive
e. $5 \cdot 3 + 0$

Answer: c

Lecture 5 Notes

CEMP005-01



Lecture 5. Order of Operations

P	Parentheses
E	Exponents
MD	Multiplication, Division
AS	Addition, Subtraction

Example 1: $5 \cdot 3^2 - 3(4 + 2 \cdot 3)$

$= 5 \cdot 3^2 - 3(4 + 6)$

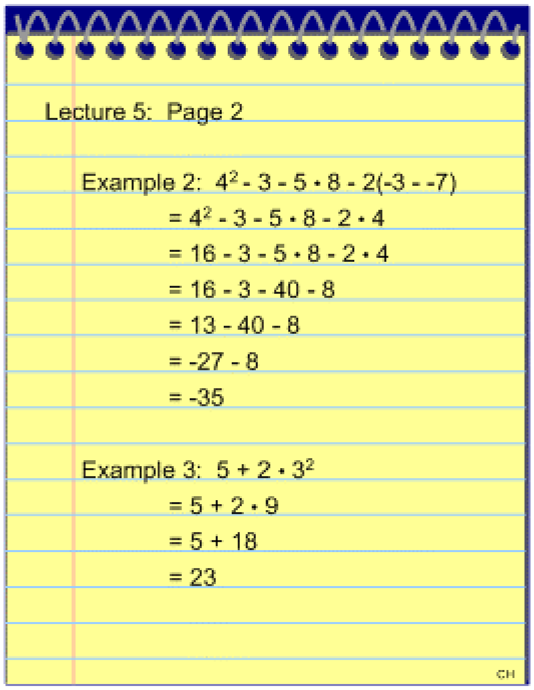
$= 5 \cdot 3^2 - 3 \cdot 10$

$= 5 \cdot 9 - 3 \cdot 10$

$= 45 - 30$

$= 15$

CEMP005-02



Lecture 5: Page 2

Example 2: $4^2 - 3 - 5 \cdot 8 - 2(-3 - -7)$

$= 4^2 - 3 - 5 \cdot 8 - 2 \cdot 4$

$= 16 - 3 - 5 \cdot 8 - 2 \cdot 4$

$= 16 - 3 - 40 - 8$

$= 13 - 40 - 8$

$= -27 - 8$

$= -35$

Example 3: $5 + 2 \cdot 3^2$

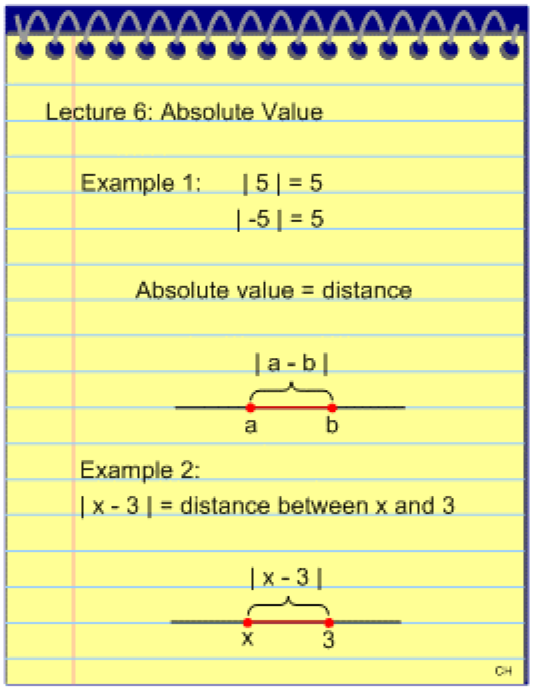
$= 5 + 2 \cdot 9$

$= 5 + 18$

$= 23$

Lecture 6 Notes

CEMP006-01



Lecture 6: Absolute Value

Example 1: $|5| = 5$

$|-5| = 5$

Absolute value = distance

$|a - b|$

a b

Example 2:

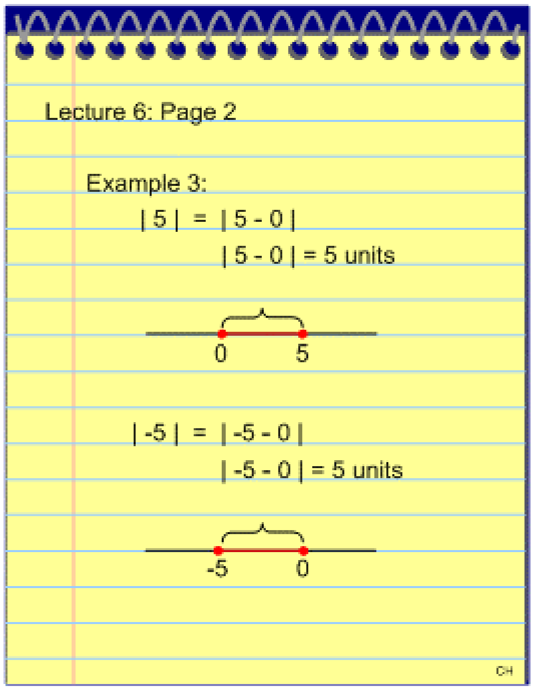
$|x - 3|$ = distance between x and 3

$|x - 3|$

x 3

CH

CEMP006-02



Lecture 6: Page 2

Example 3:

$|5| = |5 - 0|$

$|5 - 0|$ = 5 units

0 5

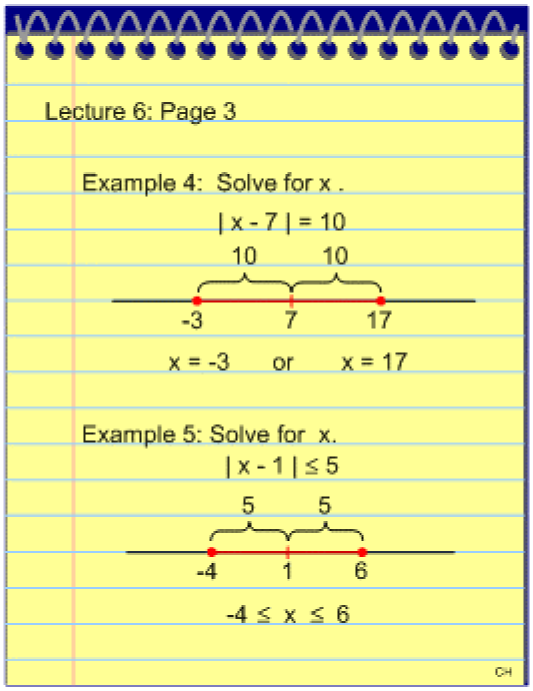
$|-5| = |-5 - 0|$

$|-5 - 0|$ = 5 units

-5 0

CH

CEMP006-03



Lecture 6: Page 3

Example 4: Solve for x .

$|x - 7| = 10$

10 10

-3 7 17

$x = -3$ or $x = 17$

Example 5: Solve for x.

$|x - 1| \leq 5$

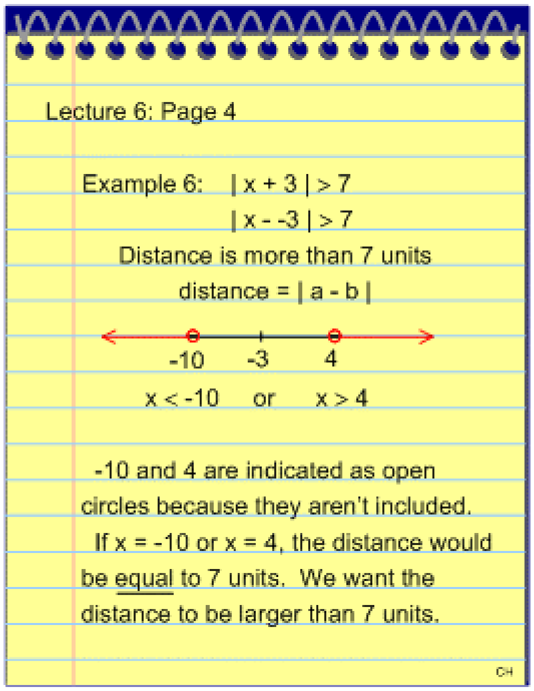
5 5

-4 1 6

$-4 \leq x \leq 6$

CH

CEMP006-04



Lecture 6: Page 4

Example 6: $|x + 3| > 7$

$|x - -3| > 7$

Distance is more than 7 units

distance = $|a - b|$

-10 -3 4

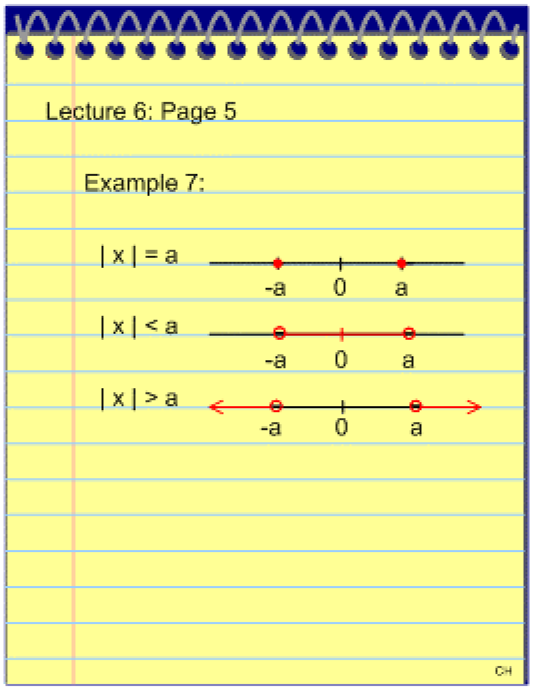
$x < -10$ or $x > 4$

-10 and 4 are indicated as open circles because they aren't included. If x = -10 or x = 4, the distance would be equal to 7 units. We want the distance to be larger than 7 units.

CH

Lecture 6 Notes, Continued

CEMP006-05



Lecture 6: Page 5

Example 7:

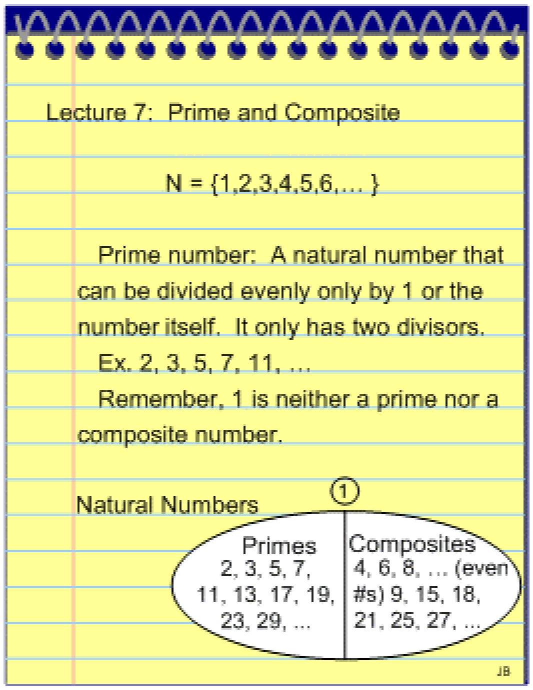
| x | = a

| x | < a

| x | > a

Lecture 7 Notes

CEMP007-01



Lecture 7: Prime and Composite

N = {1,2,3,4,5,6,… }

Prime number: A natural number that can be divided evenly only by 1 or the number itself. It only has two divisors.

Ex. 2, 3, 5, 7, 11, …

Remember, 1 is neither a prime nor a composite number.

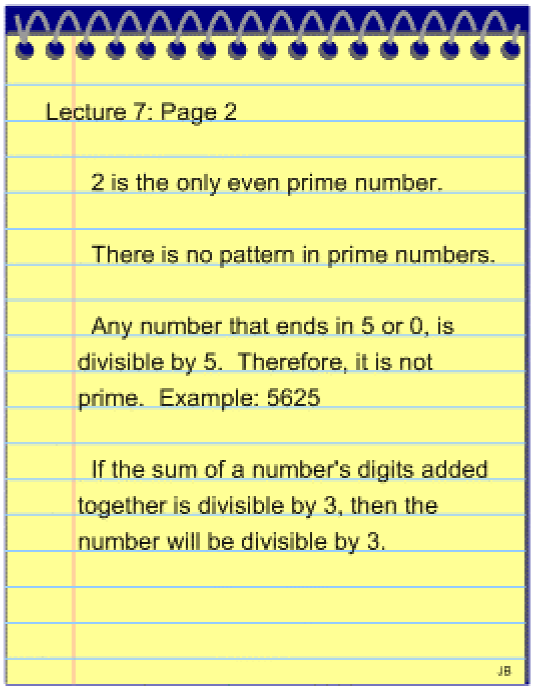
Natural Numbers

①

Primes	Composites
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …	4, 6, 8, … (even #s) 9, 15, 18, 21, 25, 27, …

JB

CEMP007-02



Lecture 7: Page 2

2 is the only even prime number.

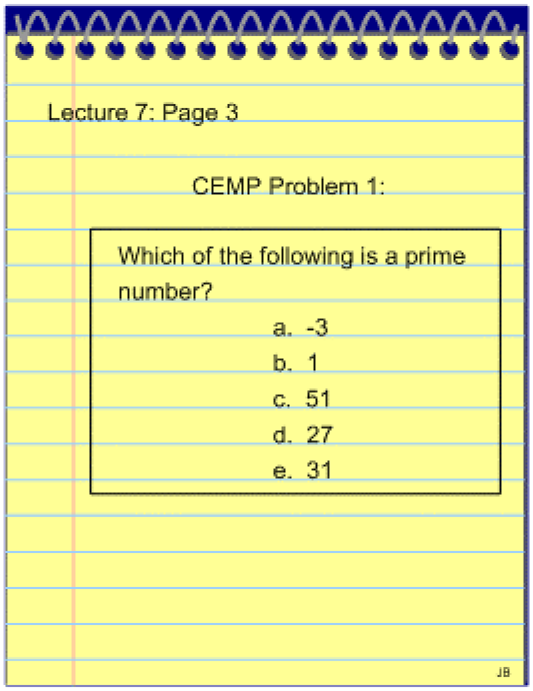
There is no pattern in prime numbers.

Any number that ends in 5 or 0, is divisible by 5. Therefore, it is not prime. Example: 5625

If the sum of a number's digits added together is divisible by 3, then the number will be divisible by 3.

JB

CEMP007-03



Lecture 7: Page 3

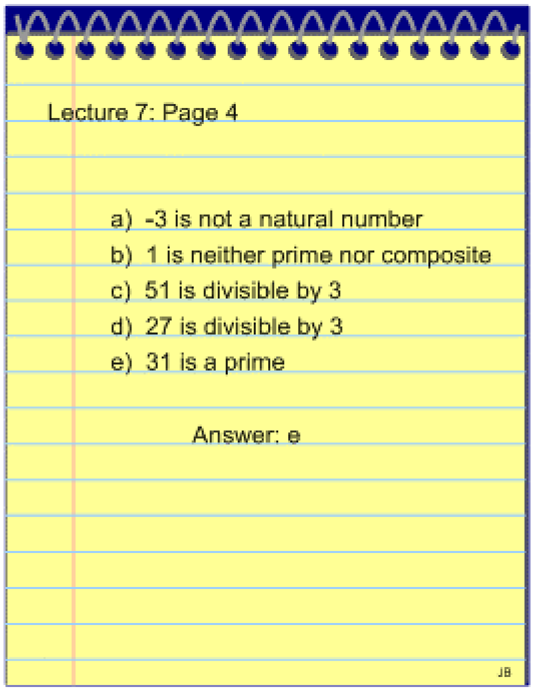
CEMP Problem 1:

Which of the following is a prime number?

a. -3
b. 1
c. 51
d. 27
e. 31

JB

CEMP007-04



Lecture 7: Page 4

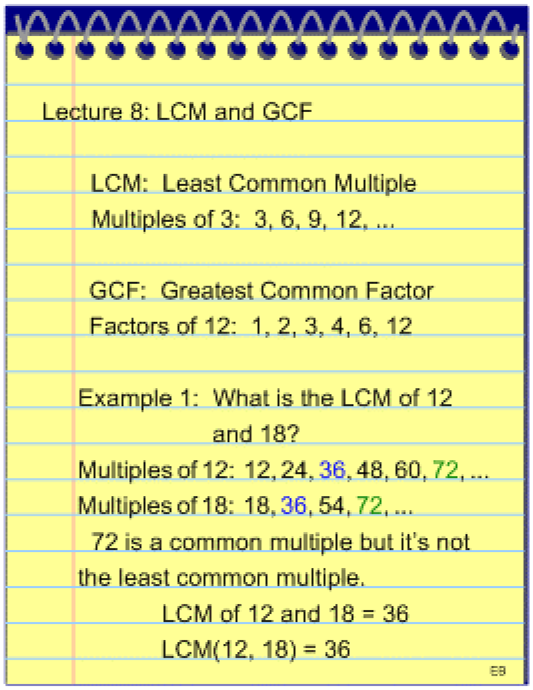
a) -3 is not a natural number
b) 1 is neither prime nor composite
c) 51 is divisible by 3
d) 27 is divisible by 3
e) 31 is a prime

Answer: e

JB

Lecture 8 Notes

CEMP008-01



Lecture 8: LCM and GCF

LCM: Least Common Multiple

Multiples of 3: 3, 6, 9, 12, ...

GCF: Greatest Common Factor

Factors of 12: 1, 2, 3, 4, 6, 12

Example 1: What is the LCM of 12 and 18?

Multiples of 12: 12, 24, 36, 48, 60, 72, ...

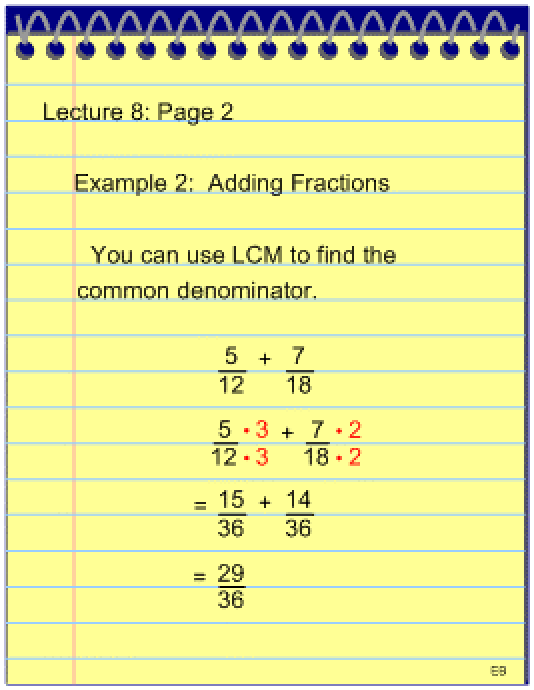
Multiples of 18: 18, 36, 54, 72, ...

72 is a common multiple but it's not the least common multiple.

LCM of 12 and 18 = 36

LCM(12, 18) = 36

CEMP008-02



Lecture 8: Page 2

Example 2: Adding Fractions

You can use LCM to find the common denominator.

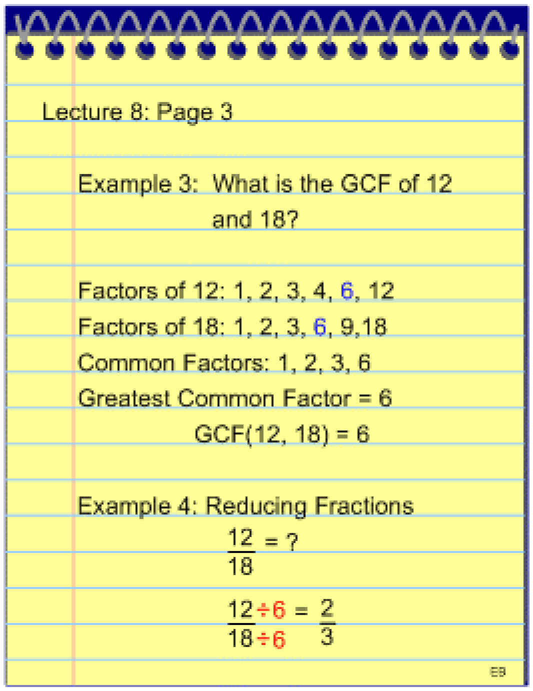
$$\frac{5}{12} + \frac{7}{18}$$

$$\frac{5 \cdot 3}{12 \cdot 3} + \frac{7 \cdot 2}{18 \cdot 2}$$

$$= \frac{15}{36} + \frac{14}{36}$$

$$= \frac{29}{36}$$

CEMP008-03



Lecture 8: Page 3

Example 3: What is the GCF of 12 and 18?

Factors of 12: 1, 2, 3, 4, 6, 12

Factors of 18: 1, 2, 3, 6, 9,18

Common Factors: 1, 2, 3, 6

Greatest Common Factor = 6

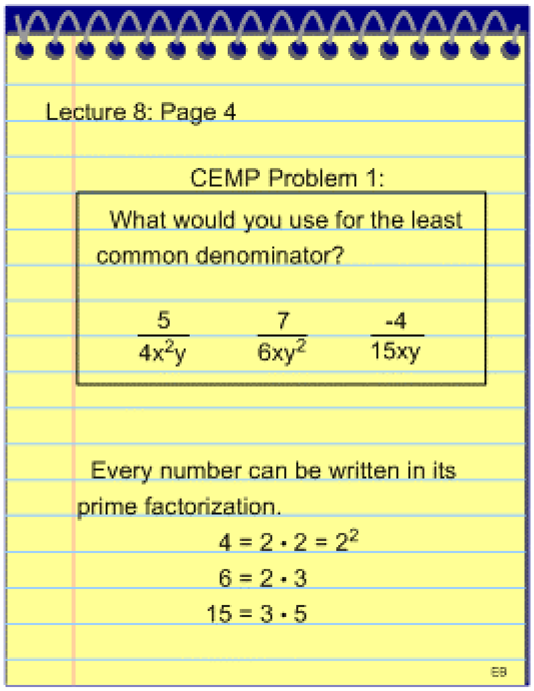
GCF(12, 18) = 6

Example 4: Reducing Fractions

$$\frac{12}{18} = ?$$

$$\frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

CEMP008-04



Lecture 8: Page 4

CEMP Problem 1:

What would you use for the least common denominator?

$$\frac{5}{4x^2y} \qquad \frac{7}{6xy^2} \qquad \frac{-4}{15xy}$$

Every number can be written in its prime factorization.

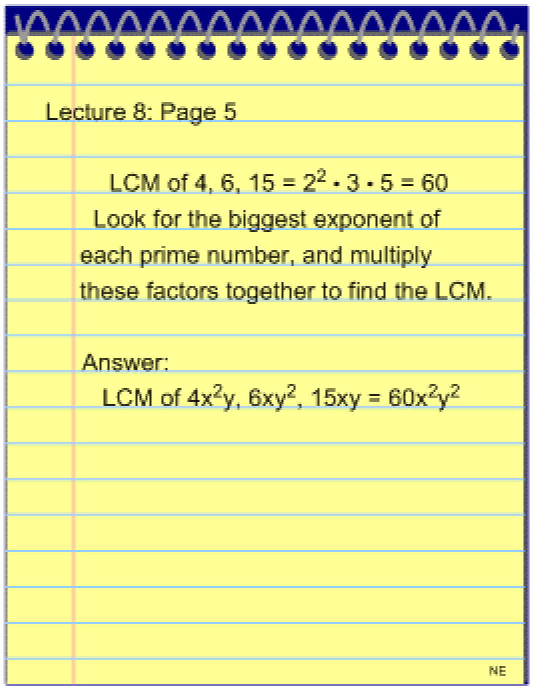
$4 = 2 \cdot 2 = 2^2$

$6 = 2 \cdot 3$

$15 = 3 \cdot 5$

Lecture 8 Notes, Continued

CEMP008-05



Lecture 8: Page 5

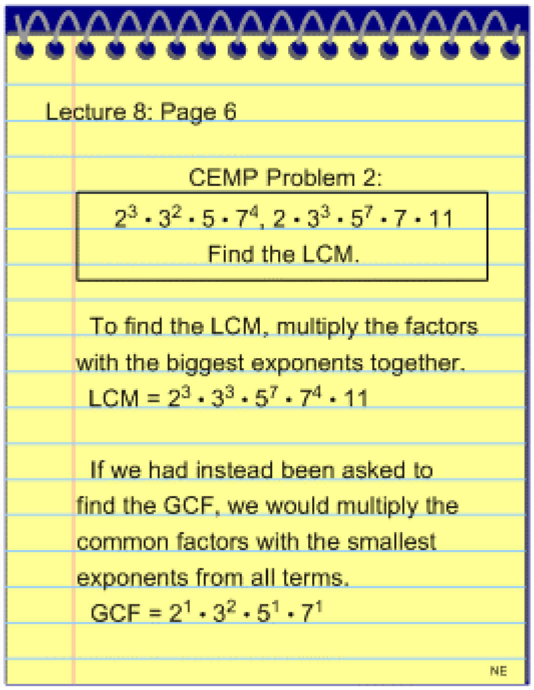
LCM of 4, 6, 15 = $2^2 \cdot 3 \cdot 5 = 60$

Look for the biggest exponent of each prime number, and multiply these factors together to find the LCM.

Answer:

LCM of $4x^2y$, $6xy^2$, $15xy = 60x^2y^2$

CEMP008-06



Lecture 8: Page 6

CEMP Problem 2:

$2^3 \cdot 3^2 \cdot 5 \cdot 7^4$, $2 \cdot 3^3 \cdot 5^7 \cdot 7 \cdot 11$

Find the LCM.

To find the LCM, multiply the factors with the biggest exponents together.

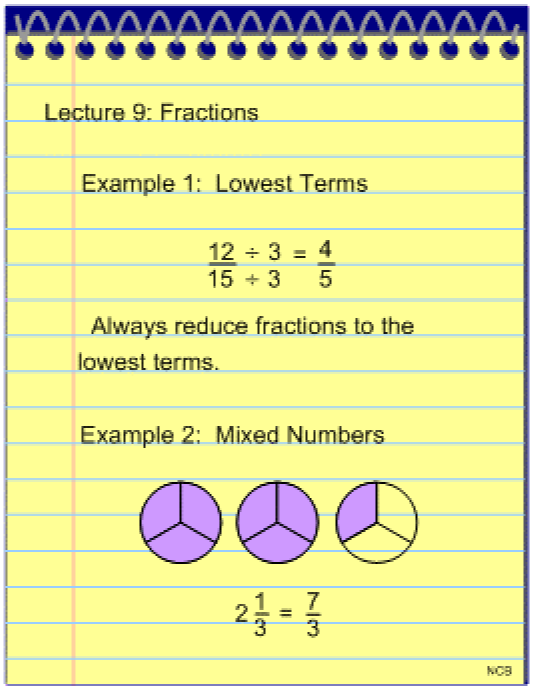
LCM = $2^3 \cdot 3^3 \cdot 5^7 \cdot 7^4 \cdot 11$

If we had instead been asked to find the GCF, we would multiply the common factors with the smallest exponents from all terms.

GCF = $2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1$

Lecture 9 Notes

CEMP009-01



Lecture 9: Fractions

Example 1: Lowest Terms

$$\frac{12 \div 3}{15 \div 3} = \frac{4}{5}$$

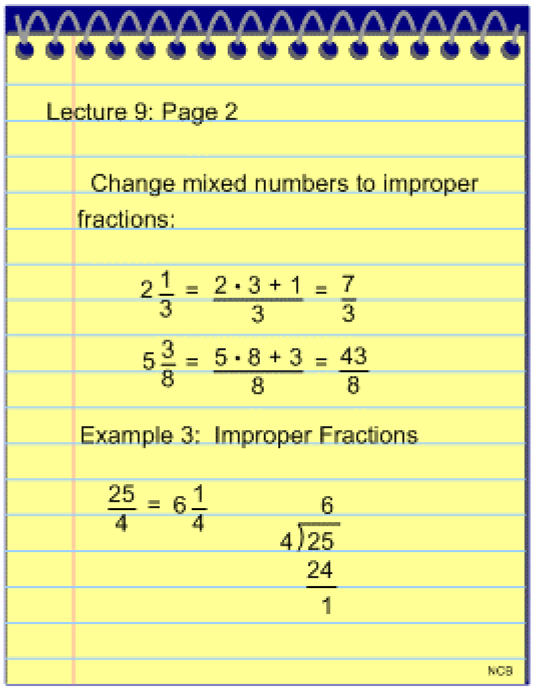
Always reduce fractions to the lowest terms.

Example 2: Mixed Numbers

$$2\frac{1}{3} = \frac{7}{3}$$

NCB

CEMP009-02



Lecture 9: Page 2

Change mixed numbers to improper fractions:

$$2\frac{1}{3} = \frac{2 \cdot 3 + 1}{3} = \frac{7}{3}$$

$$5\frac{3}{8} = \frac{5 \cdot 8 + 3}{8} = \frac{43}{8}$$

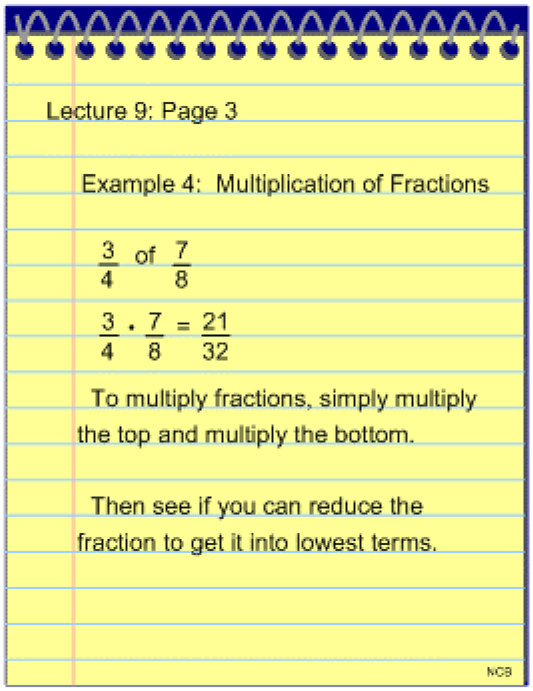
Example 3: Improper Fractions

$$\frac{25}{4} = 6\frac{1}{4}$$

$$\begin{array}{r} 6 \\ 4\overline{)25} \\ \underline{24} \\ 1 \end{array}$$

NCB

CEMP009-03



Lecture 9: Page 3

Example 4: Multiplication of Fractions

$$\frac{3}{4} \text{ of } \frac{7}{8}$$

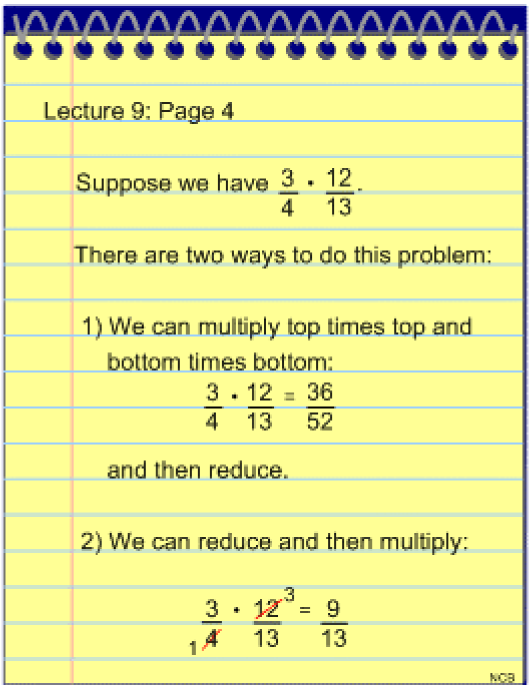
$$\frac{3}{4} \cdot \frac{7}{8} = \frac{21}{32}$$

To multiply fractions, simply multiply the top and multiply the bottom.

Then see if you can reduce the fraction to get it into lowest terms.

NCB

CEMP009-04



Lecture 9: Page 4

Suppose we have $\frac{3}{4} \cdot \frac{12}{13}$.

There are two ways to do this problem:

1) We can multiply top times top and bottom times bottom:

$$\frac{3}{4} \cdot \frac{12}{13} = \frac{36}{52}$$

and then reduce.

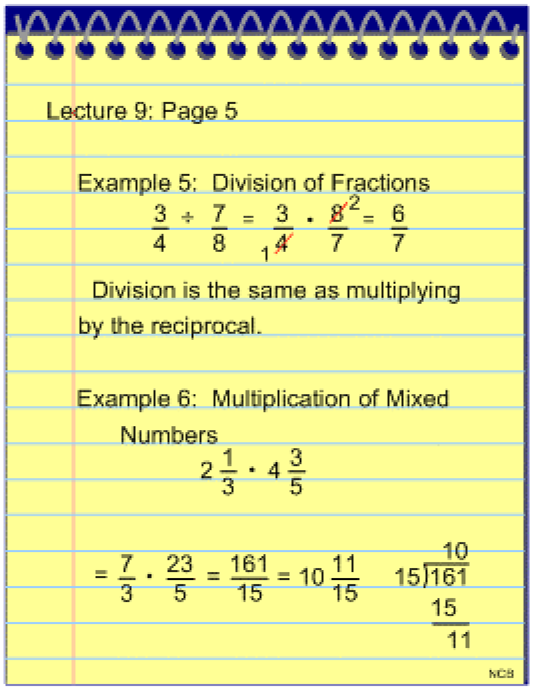
2) We can reduce and then multiply:

$$\frac{3}{{}_1\cancel{4}} \cdot \frac{\cancel{12}^{3}}{13} = \frac{9}{13}$$

NCB

Lecture 9 Notes, Continued

CEMP009-05



Lecture 9: Page 5

Example 5: Division of Fractions

$$\frac{3}{4} \div \frac{7}{8} = \frac{3}{{}_1\cancel{4}} \cdot \frac{\cancel{8}^{2}}{7} = \frac{6}{7}$$

Division is the same as multiplying by the reciprocal.

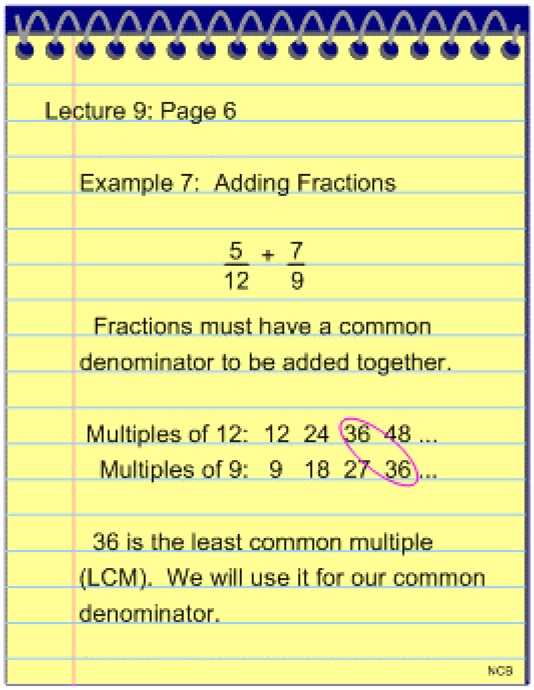
Example 6: Multiplication of Mixed Numbers

$$2\frac{1}{3} \cdot 4\frac{3}{5}$$

$$= \frac{7}{3} \cdot \frac{23}{5} = \frac{161}{15} = 10\frac{11}{15} \qquad \begin{array}{r} 10 \\ 15\overline{)161} \\ \underline{15} \\ 11 \end{array}$$

NCB

CEMP009-06



Lecture 9: Page 6

Example 7: Adding Fractions

$$\frac{5}{12} + \frac{7}{9}$$

Fractions must have a common denominator to be added together.

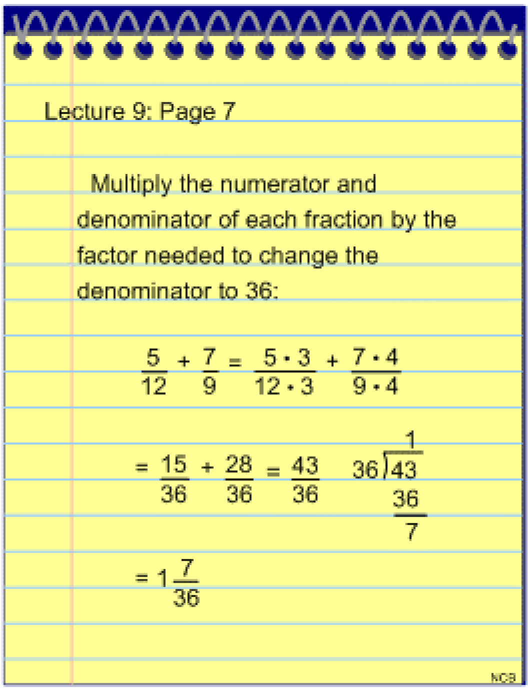
Multiples of 12: 12 24 36 48 ...

Multiples of 9: 9 18 27 36 ...

36 is the least common multiple (LCM). We will use it for our common denominator.

NCB

CEMP009-07



Lecture 9: Page 7

Multiply the numerator and denominator of each fraction by the factor needed to change the denominator to 36:

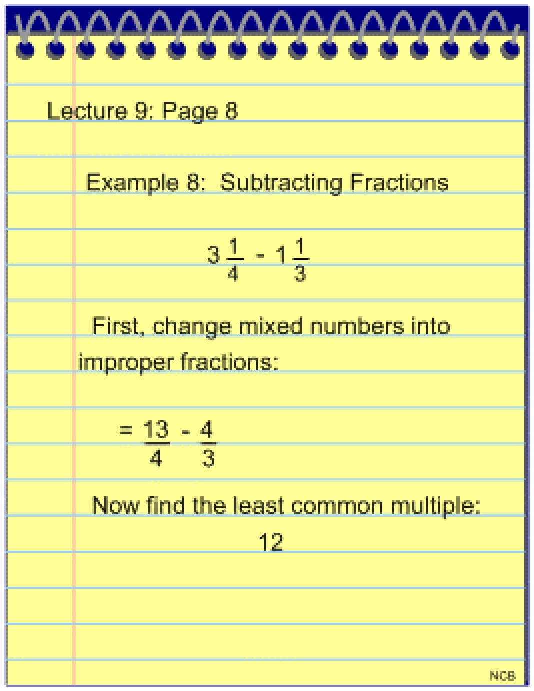
$$\frac{5}{12} + \frac{7}{9} = \frac{5 \cdot 3}{12 \cdot 3} + \frac{7 \cdot 4}{9 \cdot 4}$$

$$= \frac{15}{36} + \frac{28}{36} = \frac{43}{36} \qquad \begin{array}{r} 1 \\ 36\overline{)43} \\ \underline{36} \\ 7 \end{array}$$

$$= 1\frac{7}{36}$$

NCB

CEMP009-08



Lecture 9: Page 8

Example 8: Subtracting Fractions

$$3\frac{1}{4} - 1\frac{1}{3}$$

First, change mixed numbers into improper fractions:

$$= \frac{13}{4} - \frac{4}{3}$$

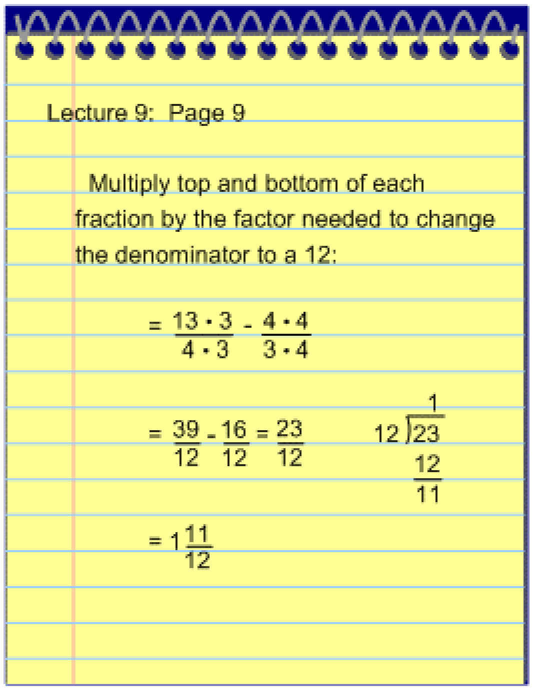
Now find the least common multiple:

12

NCB

Lecture 9 Notes, Continued

CEMP009-09



Lecture 9: Page 9

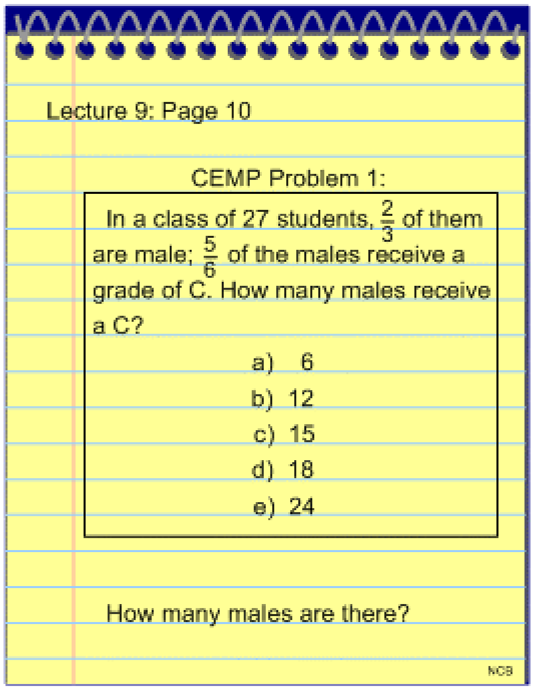
Multiply top and bottom of each fraction by the factor needed to change the denominator to a 12:

$$= \frac{13 \cdot 3}{4 \cdot 3} - \frac{4 \cdot 4}{3 \cdot 4}$$

$$= \frac{39}{12} - \frac{16}{12} = \frac{23}{12} \qquad \begin{array}{r} 1 \\ 12\overline{)23} \\ \underline{12} \\ 11 \end{array}$$

$$= 1\frac{11}{12}$$

CEMP009-10



Lecture 9: Page 10

CEMP Problem 1:

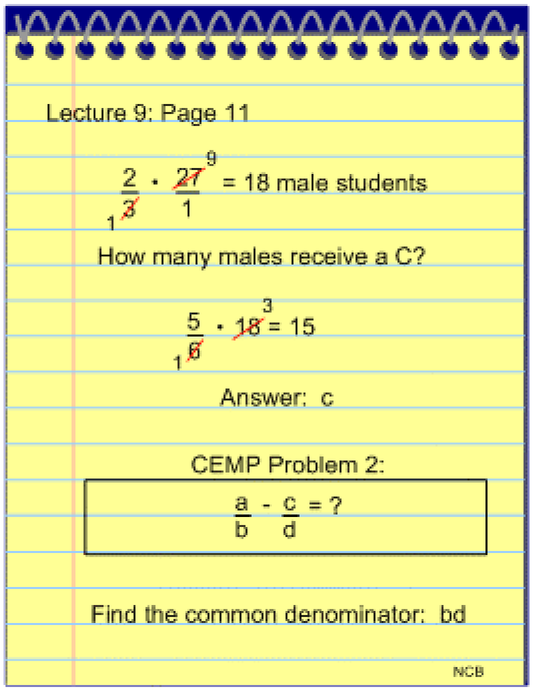
In a class of 27 students, $\frac{2}{3}$ of them are male; $\frac{5}{6}$ of the males receive a grade of C. How many males receive a C?

a) 6
b) 12
c) 15
d) 18
e) 24

How many males are there?

NCB

CEMP009-11



Lecture 9: Page 11

$$\frac{2}{{}_1\cancel{3}} \cdot \frac{\cancel{27}^{9}}{1} = 18 \text{ male students}$$

How many males receive a C?

$$\frac{5}{{}_1\cancel{6}} \cdot \cancel{18}^{3} = 15$$

Answer: c

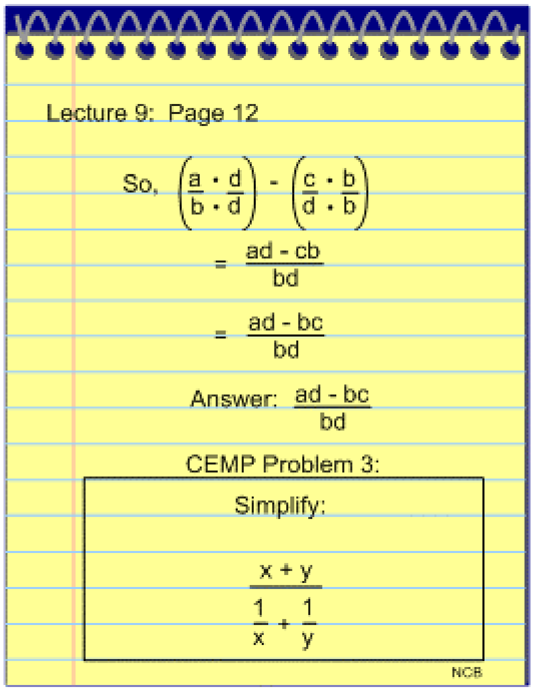
CEMP Problem 2:

$$\frac{a}{b} - \frac{c}{d} = ?$$

Find the common denominator: bd

NCB

CEMP009-12



Lecture 9: Page 12

$$\text{So, } \left(\frac{a}{b} \cdot \frac{d}{d}\right) - \left(\frac{c}{d} \cdot \frac{b}{b}\right)$$

$$= \frac{ad - cb}{bd}$$

$$= \frac{ad - bc}{bd}$$

Answer: $\frac{ad - bc}{bd}$

CEMP Problem 3:

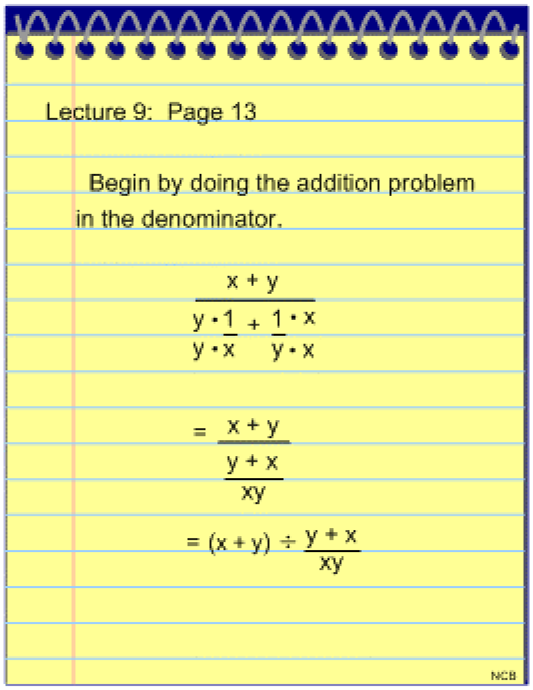
Simplify:

$$\frac{x + y}{\frac{1}{x} + \frac{1}{y}}$$

NCB

Lecture 9 Notes, Continued

CEMP009-13



Lecture 9: Page 13

Begin by doing the addition problem in the denominator.

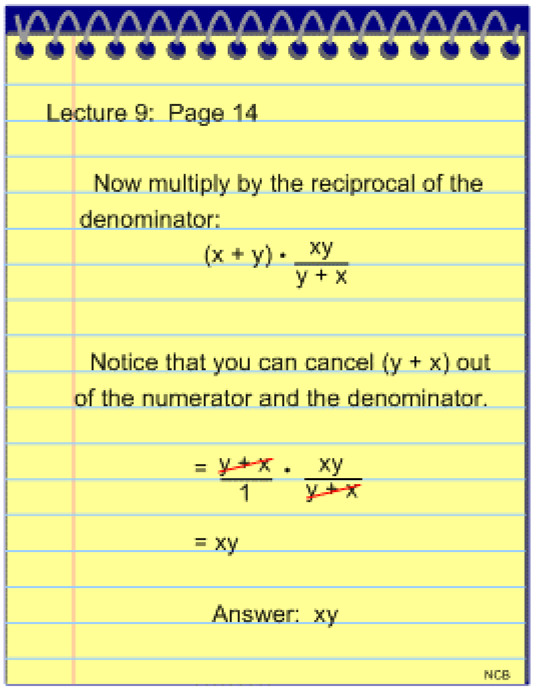
$$\frac{x+y}{\frac{y\cdot 1}{y\cdot x}+\frac{1\cdot x}{y\cdot x}}$$

$$=\frac{x+y}{\frac{y+x}{xy}}$$

$$=(x+y)\div\frac{y+x}{xy}$$

NCB

CEMP009-14



Lecture 9: Page 14

Now multiply by the reciprocal of the denominator:

$$(x+y)\cdot\frac{xy}{y+x}$$

Notice that you can cancel (y + x) out of the numerator and the denominator.

$$=\frac{\cancel{y+x}}{1}\cdot\frac{xy}{\cancel{y+x}}$$

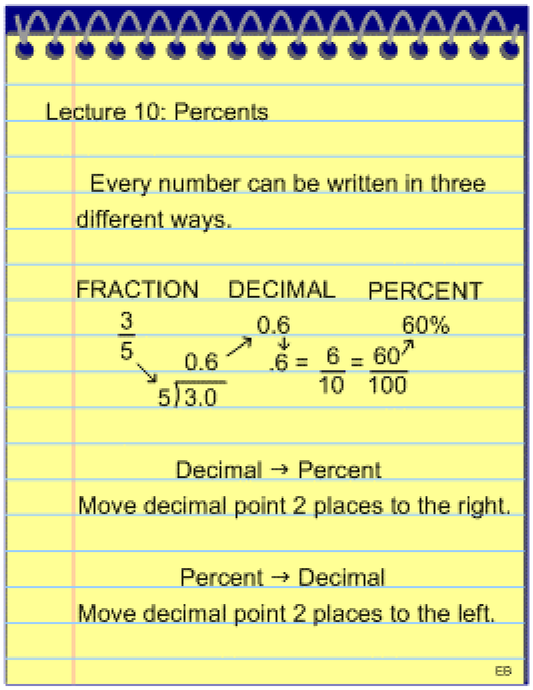
$$=xy$$

Answer: xy

NCB

Lecture 10 Notes

CEMP010-01



Lecture 10: Percents

Every number can be written in three different ways.

FRACTION DECIMAL PERCENT

$\frac{3}{5}$ $\searrow$ $5\overline{)3.0}$ with quotient 0.6 $\nearrow$ 0.6

$0.6 \downarrow .6 = \frac{6}{10} = \frac{60}{100} \nearrow 60\%$

Decimal → Percent

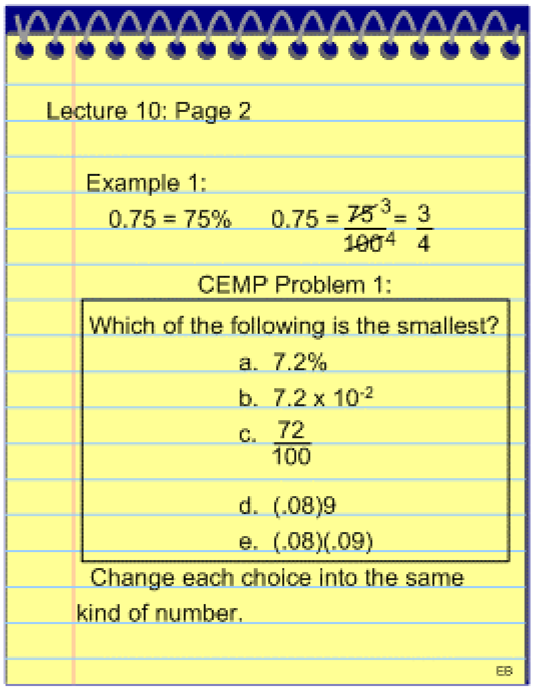
Move decimal point 2 places to the right.

Percent → Decimal

Move decimal point 2 places to the left.

EB

CEMP010-02



Lecture 10: Page 2

Example 1:

$0.75 = 75\%$ $\quad 0.75 = \frac{\cancel{75}^{3}}{\cancel{100}^{4}} = \frac{3}{4}$

CEMP Problem 1:

Which of the following is the smallest?

a. 7.2%

b. 7.2×10^{-2}

c. $\frac{72}{100}$

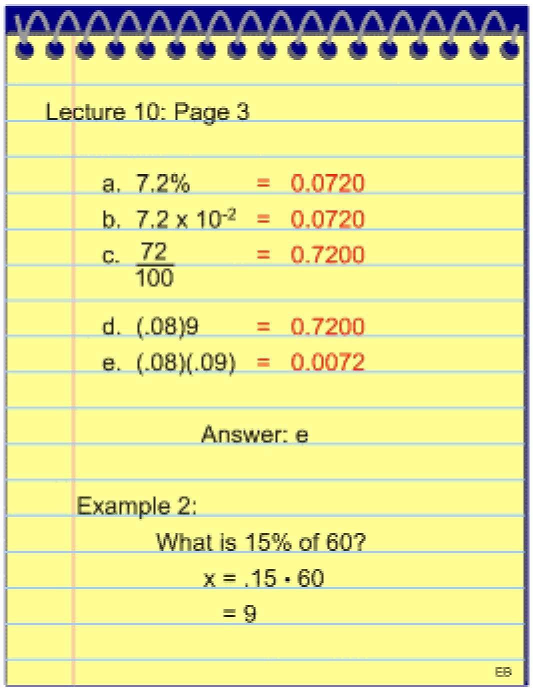
d. $(.08)9$

e. $(.08)(.09)$

Change each choice into the same kind of number.

EB

CEMP010-03



Lecture 10: Page 3

a. $7.2\% = 0.0720$

b. $7.2 \times 10^{-2} = 0.0720$

c. $\frac{72}{100} = 0.7200$

d. $(.08)9 = 0.7200$

e. $(.08)(.09) = 0.0072$

Answer: e

Example 2:

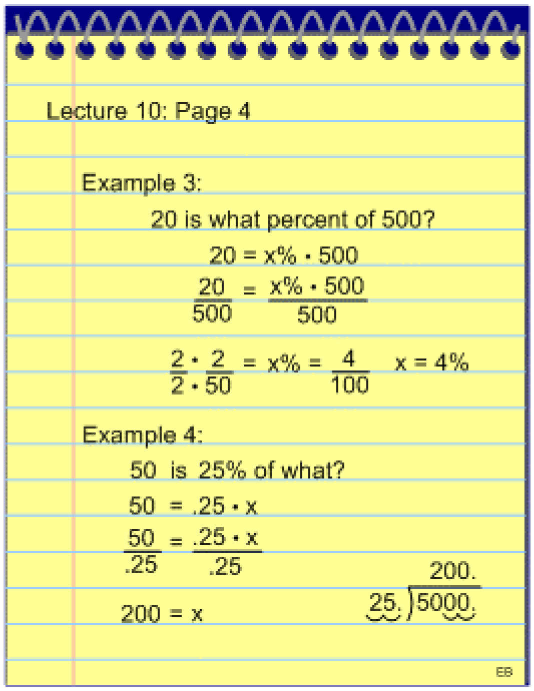
What is 15% of 60?

$x = .15 \cdot 60$

$= 9$

EB

CEMP010-04



Lecture 10: Page 4

Example 3:

20 is what percent of 500?

$20 = x\% \cdot 500$

$\frac{20}{500} = \frac{x\% \cdot 500}{500}$

$\frac{2 \cdot 2}{2 \cdot 50} = x\% = \frac{4}{100}$ $\quad x = 4\%$

Example 4:

50 is 25% of what?

$50 = .25 \cdot x$

$\frac{50}{.25} = \frac{.25 \cdot x}{.25}$

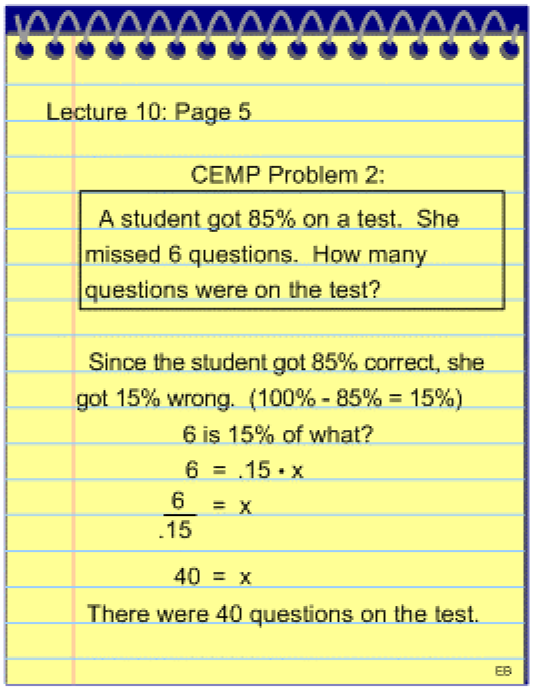
$200 = x$

$\underset{\smile}{25.}\overline{)50\underset{\smile\smile}{00.}}$ with quotient $200.$

EB

Lecture 10 Notes, Continued

CEMP010-05



Lecture 10: Page 5

CEMP Problem 2:

A student got 85% on a test. She missed 6 questions. How many questions were on the test?

Since the student got 85% correct, she got 15% wrong. (100% - 85% = 15%)

6 is 15% of what?

$$6 = .15 \cdot x$$

$$\frac{6}{.15} = x$$

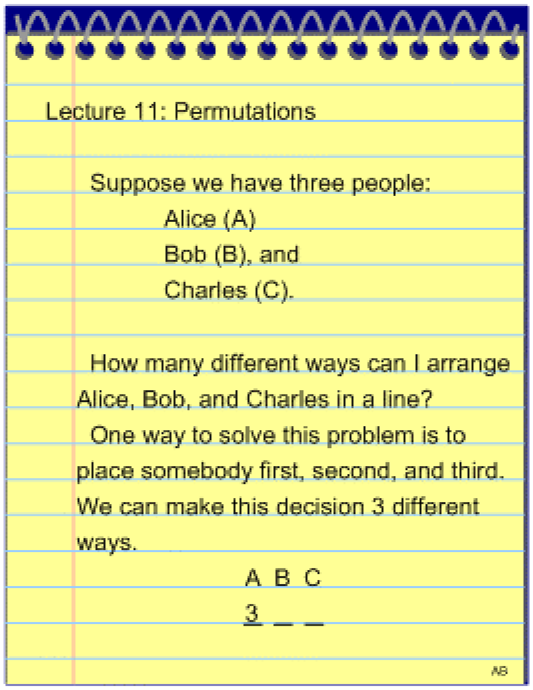
$$40 = x$$

There were 40 questions on the test.

EB

Lecture 11 Notes

CEMP011-01



Lecture 11: Permutations

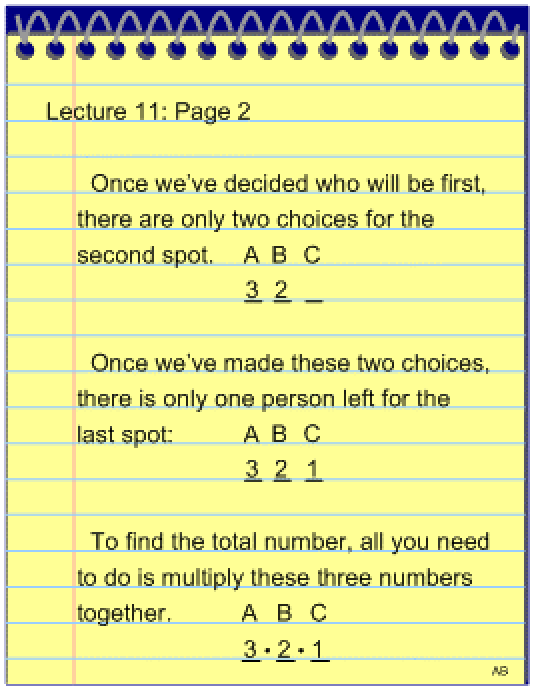
Suppose we have three people:
Alice (A)
Bob (B), and
Charles (C).

How many different ways can I arrange Alice, Bob, and Charles in a line?

One way to solve this problem is to place somebody first, second, and third. We can make this decision 3 different ways.

A B C
3 _ _

CEMP011-02



Lecture 11: Page 2

Once we've decided who will be first, there are only two choices for the second spot.

A B C
3 2 _

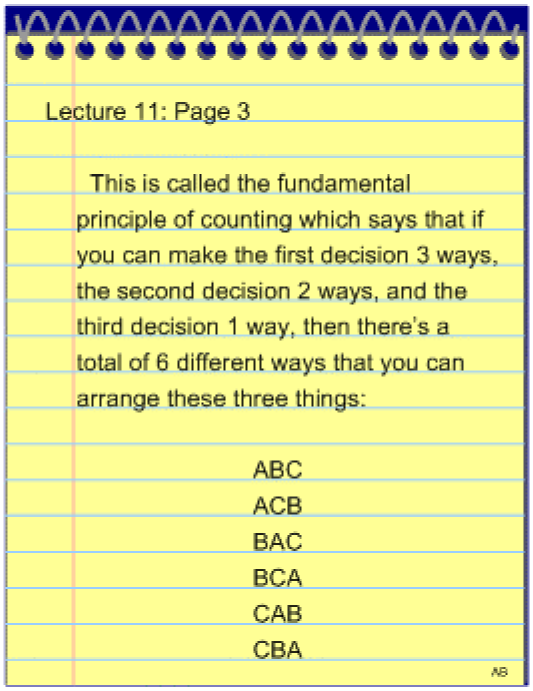
Once we've made these two choices, there is only one person left for the last spot:

A B C
3 2 1

To find the total number, all you need to do is multiply these three numbers together.

A B C
$\underline{3} \cdot \underline{2} \cdot \underline{1}$

CEMP011-03

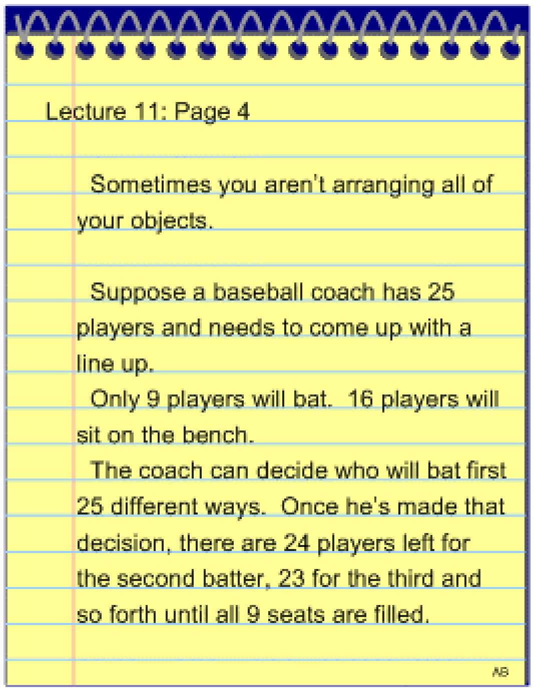


Lecture 11: Page 3

This is called the fundamental principle of counting which says that if you can make the first decision 3 ways, the second decision 2 ways, and the third decision 1 way, then there's a total of 6 different ways that you can arrange these three things:

ABC
ACB
BAC
BCA
CAB
CBA

CEMP011-04



Lecture 11: Page 4

Sometimes you aren't arranging all of your objects.

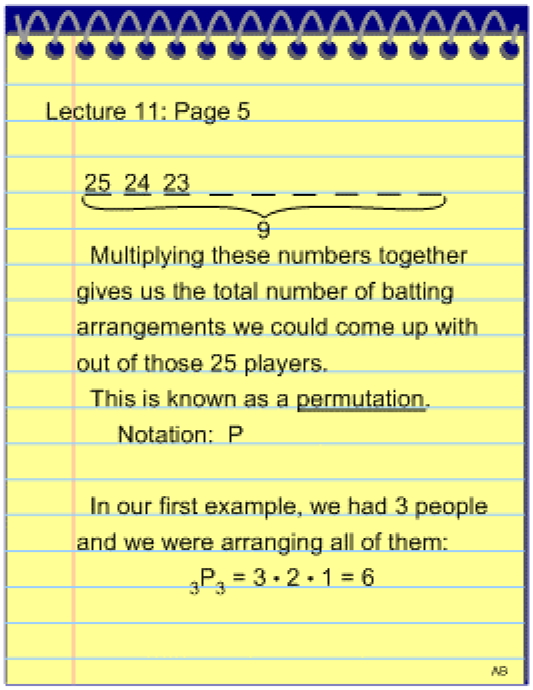
Suppose a baseball coach has 25 players and needs to come up with a line up.

Only 9 players will bat. 16 players will sit on the bench.

The coach can decide who will bat first 25 different ways. Once he's made that decision, there are 24 players left for the second batter, 23 for the third and so forth until all 9 seats are filled.

Lecture 11 Notes, Continued

CEMP011-05



Lecture 11: Page 5

$$\underbrace{\underline{25}\ \underline{24}\ \underline{23}\ \_\ \_\ \_\ \_\ \_\ \_}_{9}$$

Multiplying these numbers together gives us the total number of batting arrangements we could come up with out of those 25 players.

This is known as a permutation.

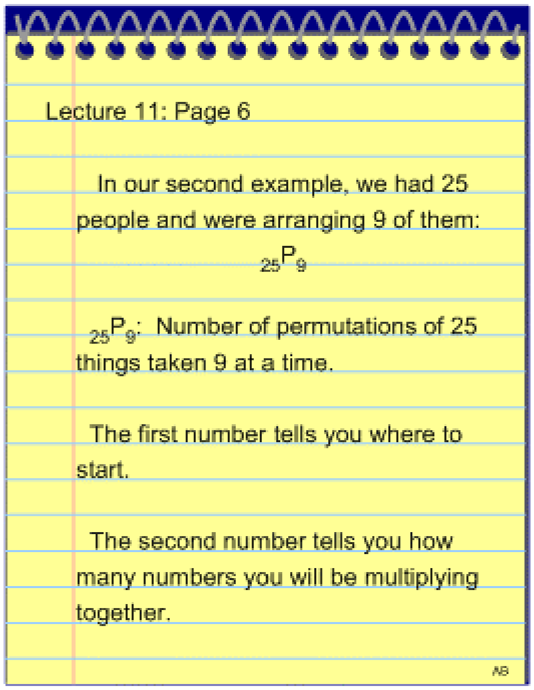
Notation: P

In our first example, we had 3 people and we were arranging all of them:

$$_3P_3 = 3 \cdot 2 \cdot 1 = 6$$

AB

CEMP011-06



Lecture 11: Page 6

In our second example, we had 25 people and were arranging 9 of them:

$$_{25}P_9$$

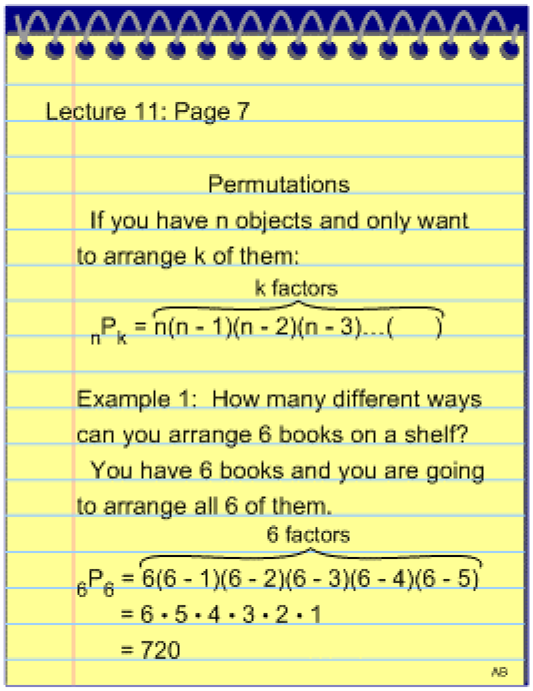
$_{25}P_9$: Number of permutations of 25 things taken 9 at a time.

The first number tells you where to start.

The second number tells you how many numbers you will be multiplying together.

AB

CEMP011-07



Lecture 11: Page 7

Permutations

If you have n objects and only want to arrange k of them:

$$_nP_k = \overbrace{n(n - 1)(n - 2)(n - 3)\ldots(\quad)}^{\text{k factors}}$$

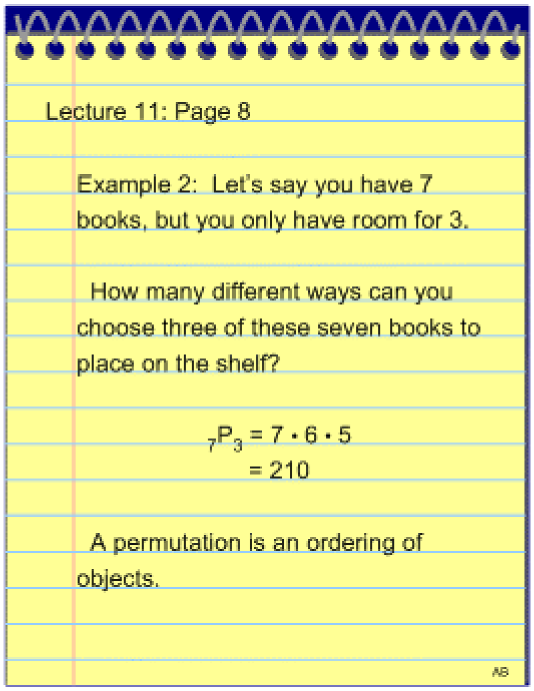
Example 1: How many different ways can you arrange 6 books on a shelf?

You have 6 books and you are going to arrange all 6 of them.

$$_6P_6 = \overbrace{6(6 - 1)(6 - 2)(6 - 3)(6 - 4)(6 - 5)}^{\text{6 factors}}$$
$$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720$$

AB

CEMP011-08



Lecture 11: Page 8

Example 2: Let's say you have 7 books, but you only have room for 3.

How many different ways can you choose three of these seven books to place on the shelf?

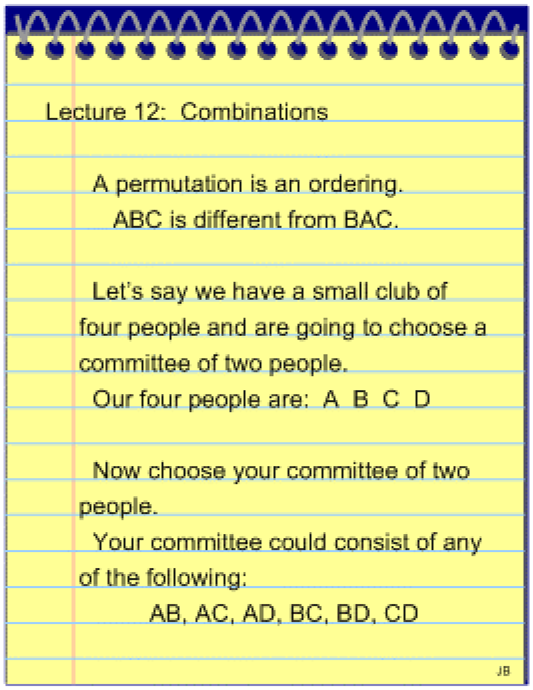
$$_7P_3 = 7 \cdot 6 \cdot 5$$
$$= 210$$

A permutation is an ordering of objects.

AB

Lecture 12 Notes

CEMP012-01



Lecture 12: Combinations

A permutation is an ordering.
ABC is different from BAC.

Let's say we have a small club of four people and are going to choose a committee of two people.

Our four people are: A B C D

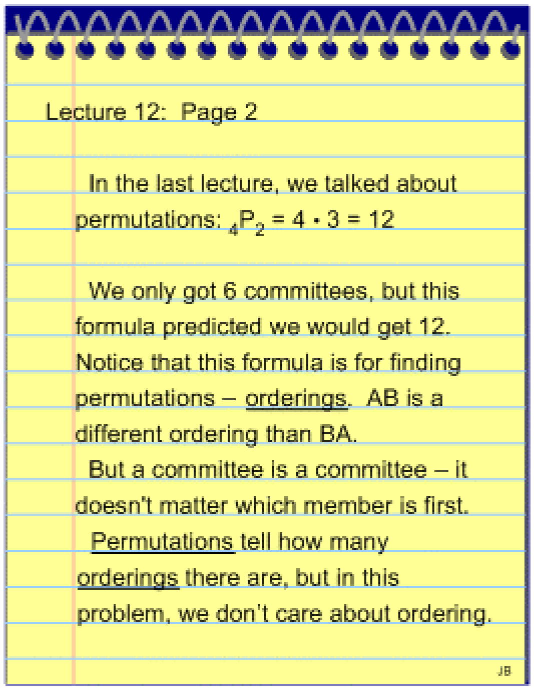
Now choose your committee of two people.

Your committee could consist of any of the following:

AB, AC, AD, BC, BD, CD

JB

CEMP012-02



Lecture 12: Page 2

In the last lecture, we talked about permutations: ${}_4P_2 = 4 \cdot 3 = 12$

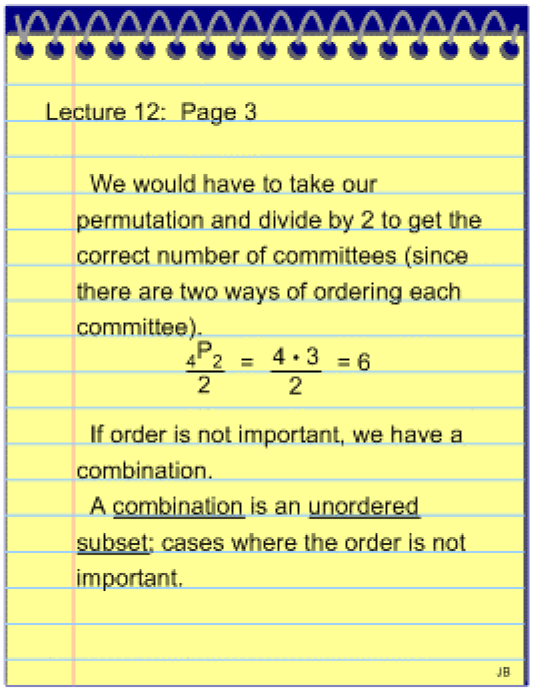
We only got 6 committees, but this formula predicted we would get 12. Notice that this formula is for finding permutations – orderings. AB is a different ordering than BA.

But a committee is a committee – it doesn't matter which member is first.

Permutations tell how many orderings there are, but in this problem, we don't care about ordering.

JB

CEMP012-03



Lecture 12: Page 3

We would have to take our permutation and divide by 2 to get the correct number of committees (since there are two ways of ordering each committee).

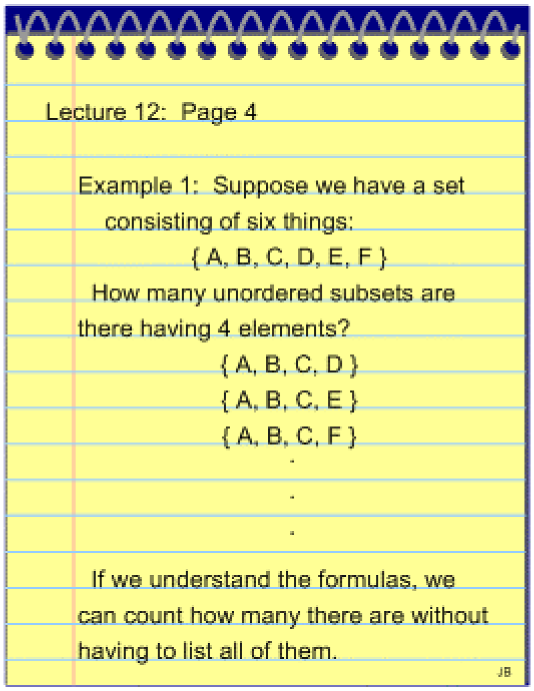
$$\frac{{}_4P_2}{2} = \frac{4 \cdot 3}{2} = 6$$

If order is not important, we have a combination.

A combination is an unordered subset; cases where the order is not important.

JB

CEMP012-04



Lecture 12: Page 4

Example 1: Suppose we have a set consisting of six things:

{ A, B, C, D, E, F }

How many unordered subsets are there having 4 elements?

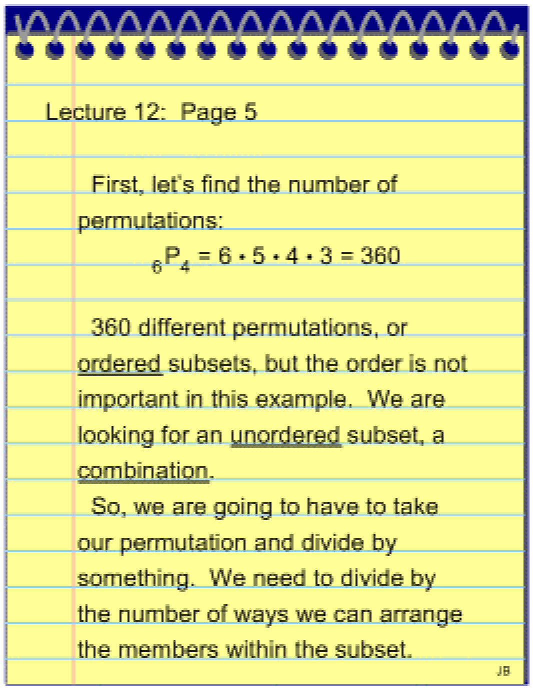
{ A, B, C, D }
{ A, B, C, E }
{ A, B, C, F }
.
.
.

If we understand the formulas, we can count how many there are without having to list all of them.

JB

Lecture 12 Notes, Continued

CEMP012-05



Lecture 12: Page 5

First, let's find the number of permutations:

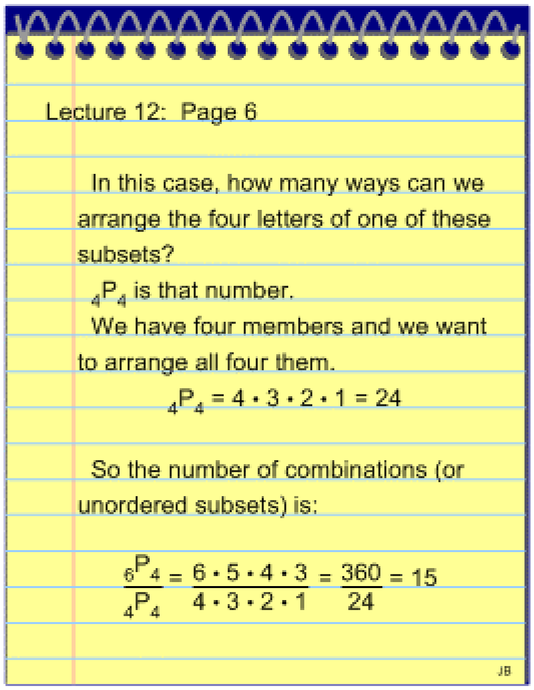
$${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

360 different permutations, or ordered subsets, but the order is not important in this example. We are looking for an unordered subset, a combination.

So, we are going to have to take our permutation and divide by something. We need to divide by the number of ways we can arrange the members within the subset.

JB

CEMP012-06



Lecture 12: Page 6

In this case, how many ways can we arrange the four letters of one of these subsets?

${}_4P_4$ is that number.

We have four members and we want to arrange all four them.

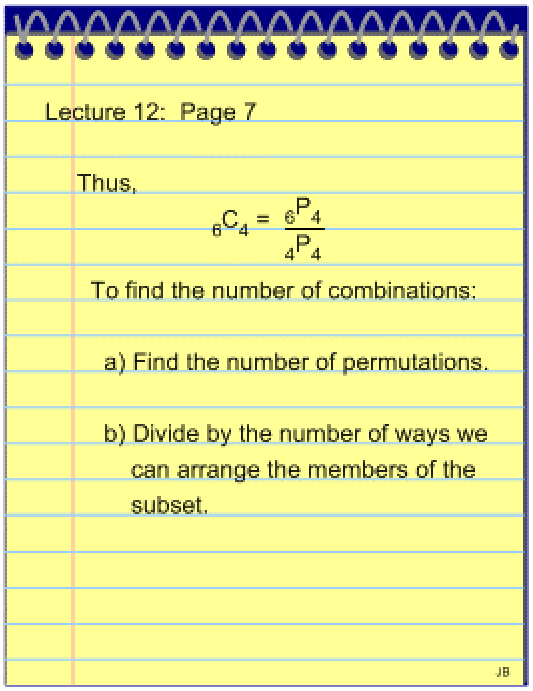
$${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

So the number of combinations (or unordered subsets) is:

$$\frac{{}_6P_4}{{}_4P_4} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{360}{24} = 15$$

JB

CEMP012-07



Lecture 12: Page 7

Thus,

$${}_6C_4 = \frac{{}_6P_4}{{}_4P_4}$$

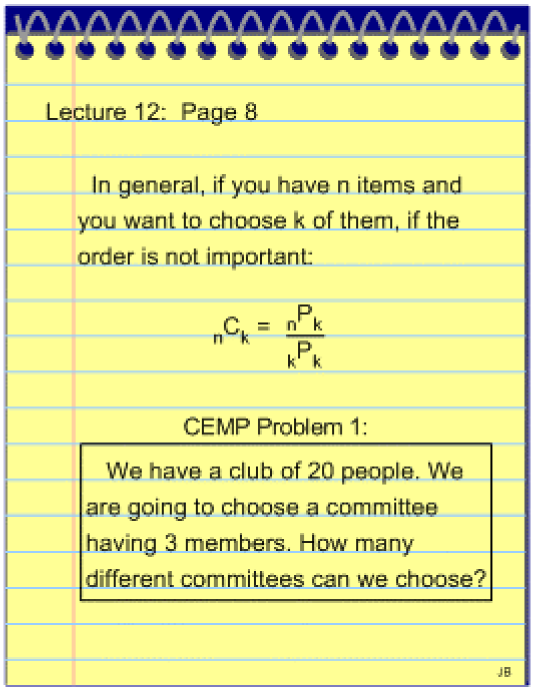
To find the number of combinations:

a) Find the number of permutations.

b) Divide by the number of ways we can arrange the members of the subset.

JB

CEMP012-08



Lecture 12: Page 8

In general, if you have n items and you want to choose k of them, if the order is not important:

$${}_nC_k = \frac{{}_nP_k}{{}_kP_k}$$

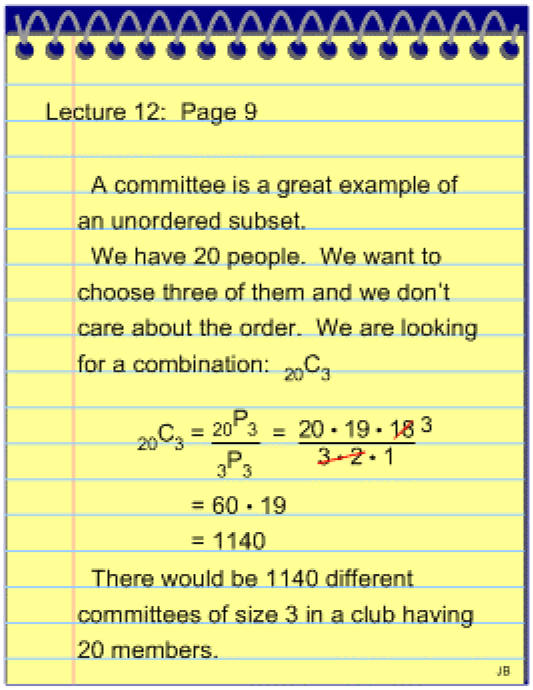
CEMP Problem 1:

We have a club of 20 people. We are going to choose a committee having 3 members. How many different committees can we choose?

JB

Lecture 12 Notes, Continued

CEMP012-09



Lecture 12: Page 9

A committee is a great example of an unordered subset.

We have 20 people. We want to choose three of them and we don't care about the order. We are looking for a combination: ${}_{20}C_3$

$${}_{20}C_3 = \frac{{}_{20}P_3}{{}_3P_3} = \frac{20 \cdot 19 \cdot \cancel{18}\,3}{\cancel{3 \cdot 2} \cdot 1}$$

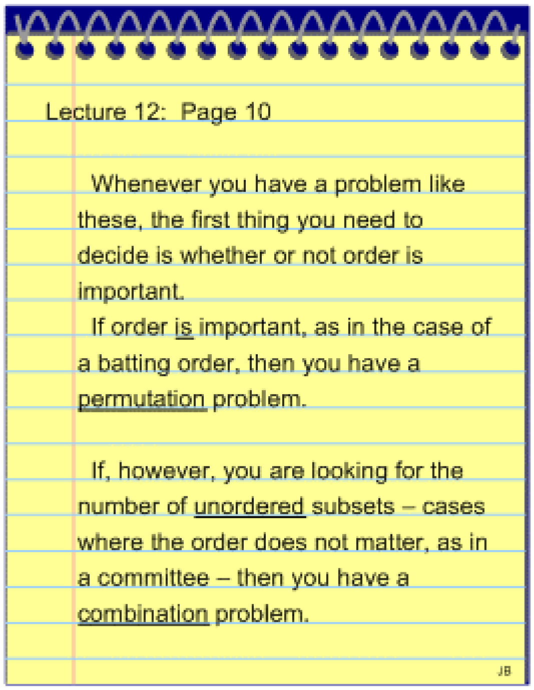
$$= 60 \cdot 19$$

$$= 1140$$

There would be 1140 different committees of size 3 in a club having 20 members.

JB

CEMP012-10



Lecture 12: Page 10

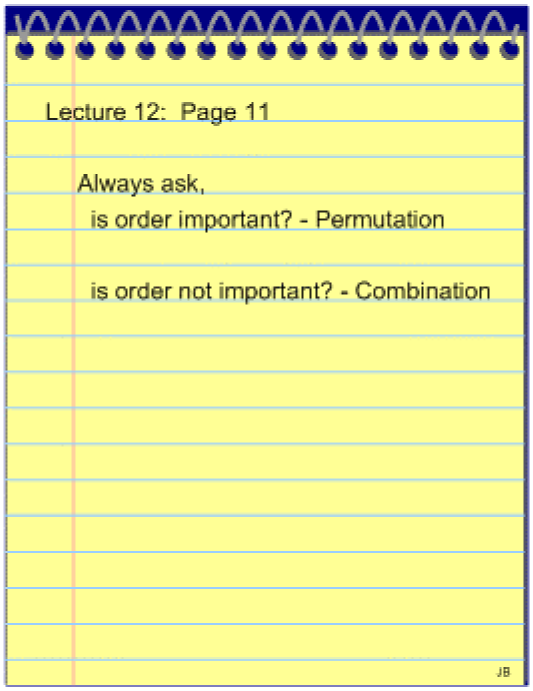
Whenever you have a problem like these, the first thing you need to decide is whether or not order is important.

If order is important, as in the case of a batting order, then you have a permutation problem.

If, however, you are looking for the number of unordered subsets – cases where the order does not matter, as in a committee – then you have a combination problem.

JB

CEMP012-11



Lecture 12: Page 11

Always ask,

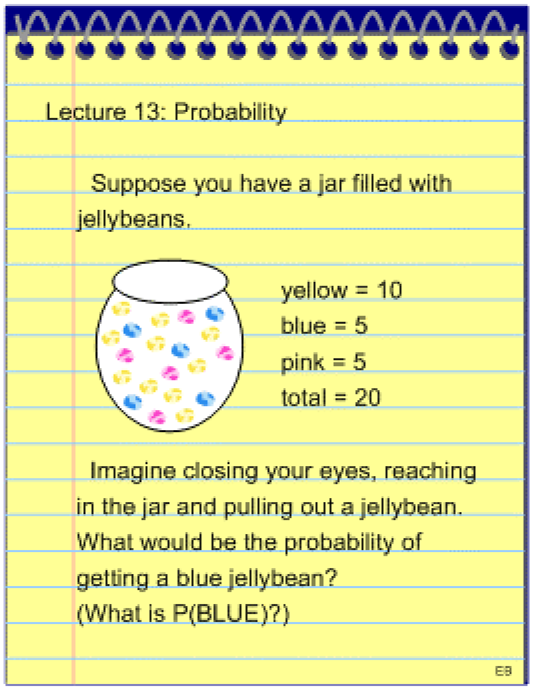
is order important? - Permutation

is order not important? - Combination

JB

Lecture 13 Notes

CEMP013-01



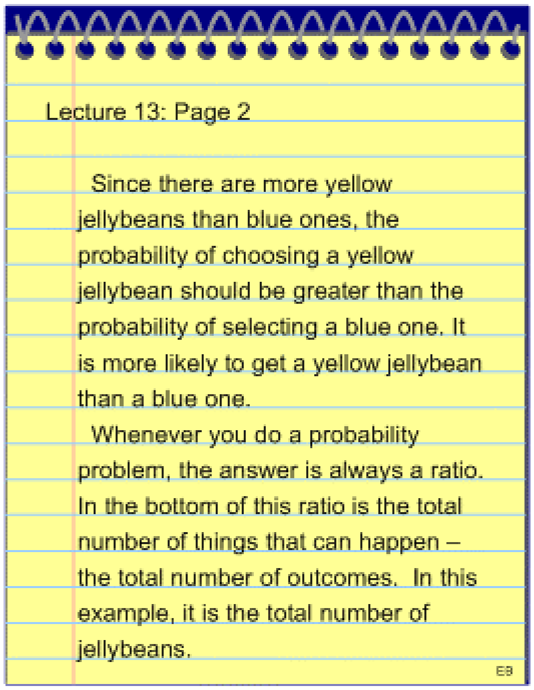
Lecture 13: Probability

Suppose you have a jar filled with jellybeans.

yellow = 10
blue = 5
pink = 5
total = 20

Imagine closing your eyes, reaching in the jar and pulling out a jellybean. What would be the probability of getting a blue jellybean?
(What is P(BLUE)?)

CEMP013-02

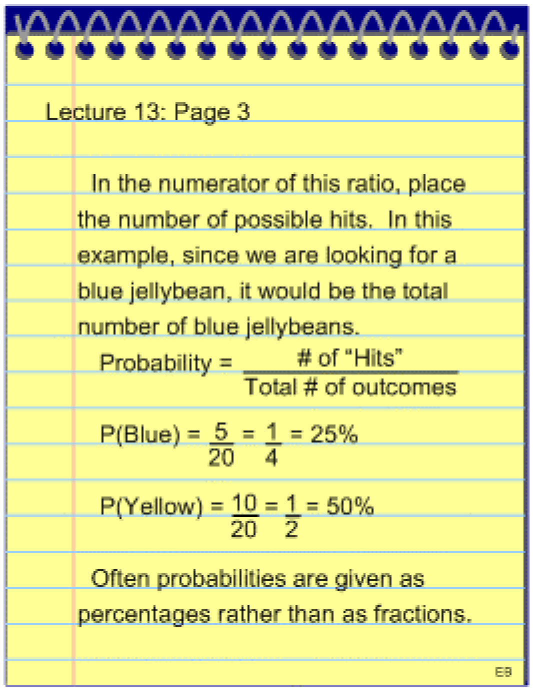


Lecture 13: Page 2

Since there are more yellow jellybeans than blue ones, the probability of choosing a yellow jellybean should be greater than the probability of selecting a blue one. It is more likely to get a yellow jellybean than a blue one.

Whenever you do a probability problem, the answer is always a ratio. In the bottom of this ratio is the total number of things that can happen – the total number of outcomes. In this example, it is the total number of jellybeans.

CEMP013-03



Lecture 13: Page 3

In the numerator of this ratio, place the number of possible hits. In this example, since we are looking for a blue jellybean, it would be the total number of blue jellybeans.

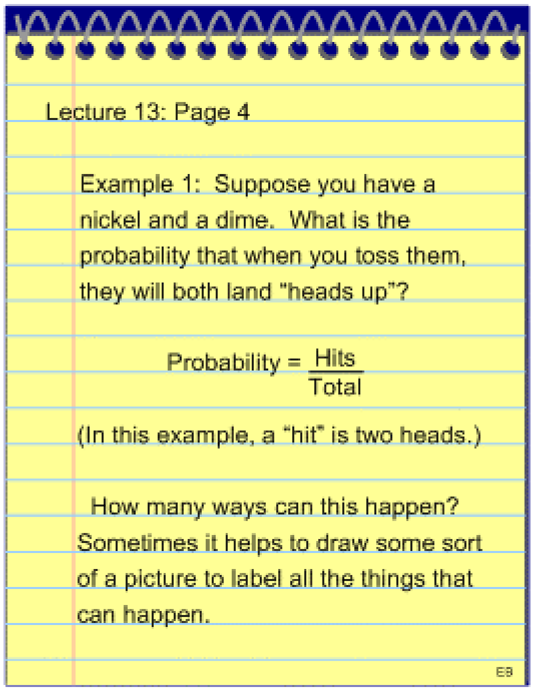
$$\text{Probability} = \frac{\text{\# of "Hits"}}{\text{Total \# of outcomes}}$$

$$P(\text{Blue}) = \frac{5}{20} = \frac{1}{4} = 25\%$$

$$P(\text{Yellow}) = \frac{10}{20} = \frac{1}{2} = 50\%$$

Often probabilities are given as percentages rather than as fractions.

CEMP013-04



Lecture 13: Page 4

Example 1: Suppose you have a nickel and a dime. What is the probability that when you toss them, they will both land "heads up"?

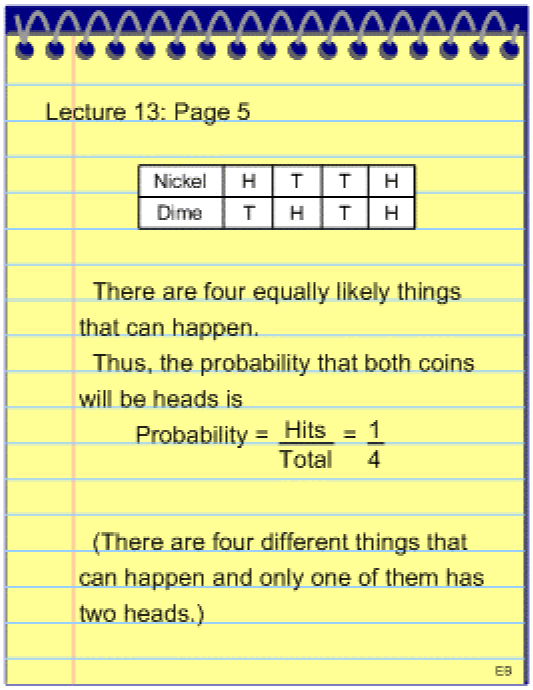
$$\text{Probability} = \frac{\text{Hits}}{\text{Total}}$$

(In this example, a "hit" is two heads.)

How many ways can this happen? Sometimes it helps to draw some sort of a picture to label all the things that can happen.

Lecture 13 Notes, Continued

CEMP013-05



Lecture 13: Page 5

Nickel	H	T	T	H
Dime	T	H	T	H

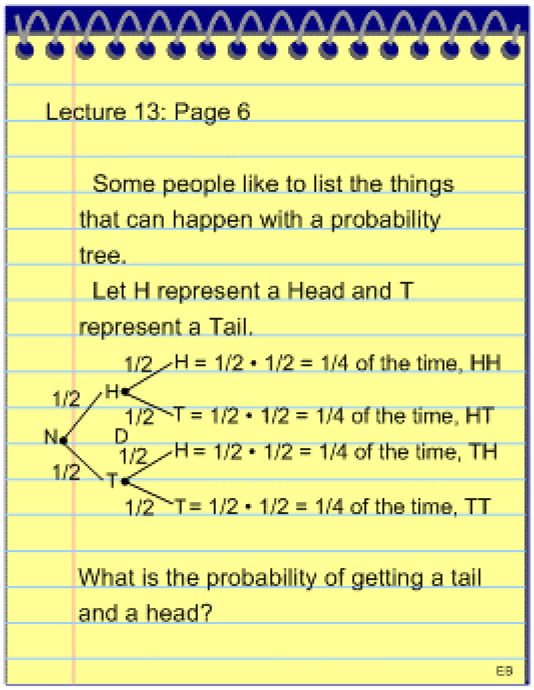
There are four equally likely things that can happen.

Thus, the probability that both coins will be heads is

$$\text{Probability} = \frac{\text{Hits}}{\text{Total}} = \frac{1}{4}$$

(There are four different things that can happen and only one of them has two heads.)

CEMP013-06



Lecture 13: Page 6

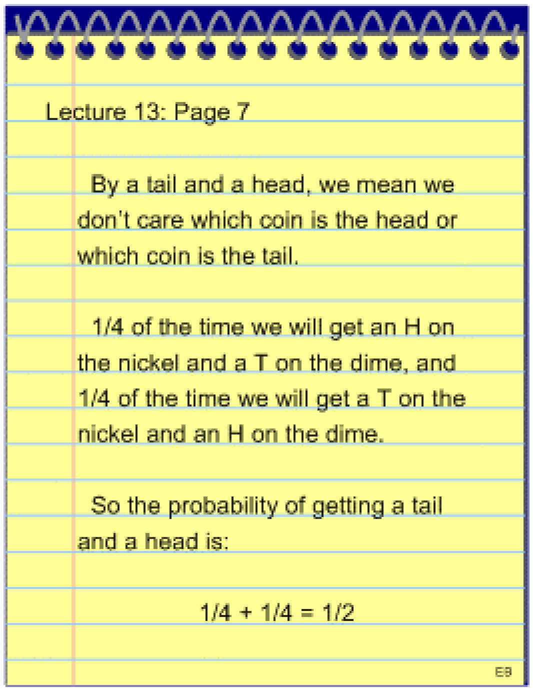
Some people like to list the things that can happen with a probability tree.

Let H represent a Head and T represent a Tail.

1/2 H = 1/2 • 1/2 = 1/4 of the time, HH
1/2 T = 1/2 • 1/2 = 1/4 of the time, HT
1/2 H = 1/2 • 1/2 = 1/4 of the time, TH
1/2 T = 1/2 • 1/2 = 1/4 of the time, TT

What is the probability of getting a tail and a head?

CEMP013-07



Lecture 13: Page 7

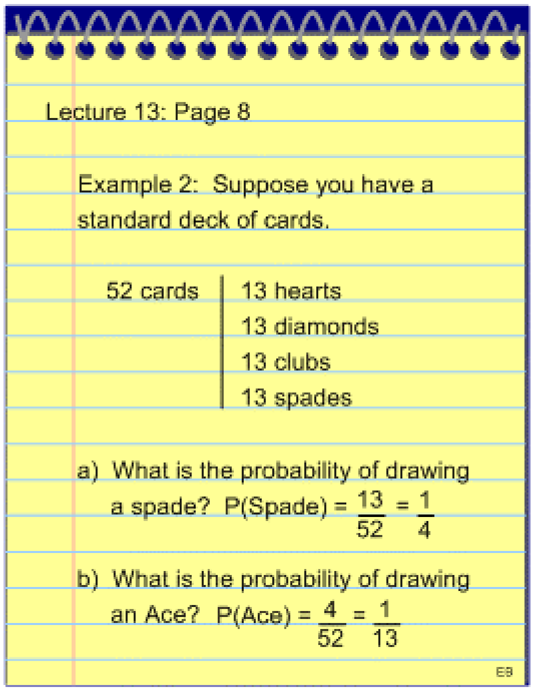
By a tail and a head, we mean we don't care which coin is the head or which coin is the tail.

1/4 of the time we will get an H on the nickel and a T on the dime, and 1/4 of the time we will get a T on the nickel and an H on the dime.

So the probability of getting a tail and a head is:

1/4 + 1/4 = 1/2

CEMP013-08



Lecture 13: Page 8

Example 2: Suppose you have a standard deck of cards.

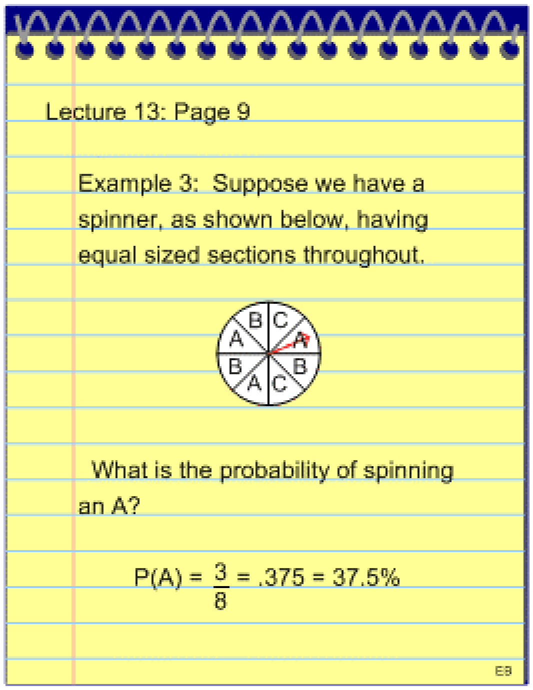
52 cards	13 hearts 13 diamonds 13 clubs 13 spades

a) What is the probability of drawing a spade? P(Spade) = $\frac{13}{52} = \frac{1}{4}$

b) What is the probability of drawing an Ace? P(Ace) = $\frac{4}{52} = \frac{1}{13}$

Lecture 13 Notes, Continued

CEMP013-09



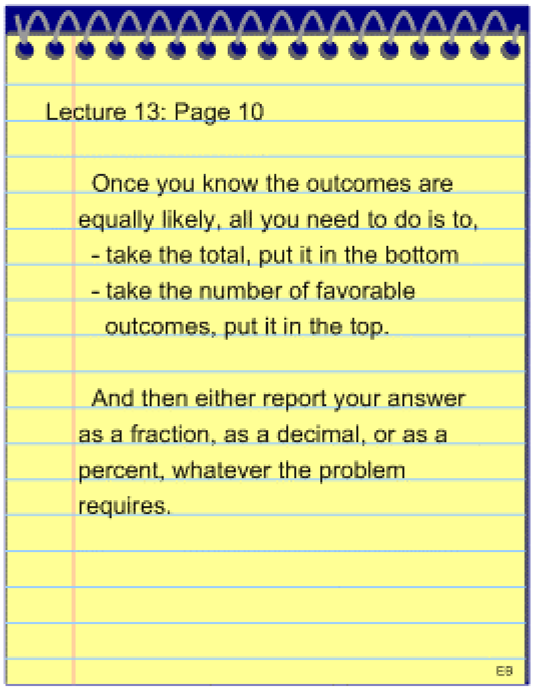
Lecture 13: Page 9

Example 3: Suppose we have a spinner, as shown below, having equal sized sections throughout.

What is the probability of spinning an A?

$$P(A) = \frac{3}{8} = .375 = 37.5\%$$

CEMP013-10



Lecture 13: Page 10

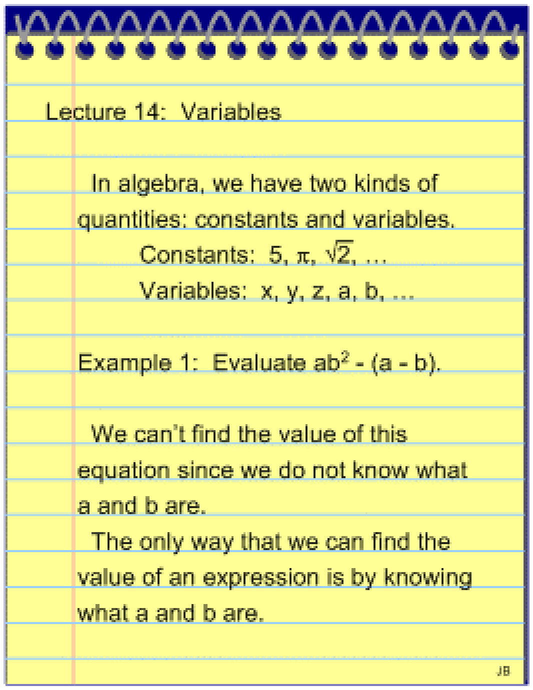
Once you know the outcomes are equally likely, all you need to do is to,

- take the total, put it in the bottom
- take the number of favorable outcomes, put it in the top.

And then either report your answer as a fraction, as a decimal, or as a percent, whatever the problem requires.

Lecture 14 Notes

CEMP014-01



Lecture 14: Variables

In algebra, we have two kinds of quantities: constants and variables.

Constants: 5, π, $\sqrt{2}$, …

Variables: x, y, z, a, b, …

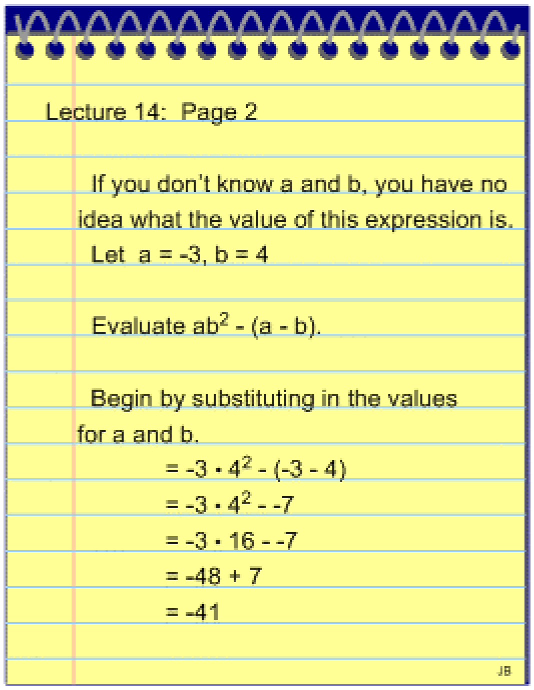
Example 1: Evaluate $ab^2 - (a - b)$.

We can't find the value of this equation since we do not know what a and b are.

The only way that we can find the value of an expression is by knowing what a and b are.

JB

CEMP014-02



Lecture 14: Page 2

If you don't know a and b, you have no idea what the value of this expression is.

Let $a = -3$, $b = 4$

Evaluate $ab^2 - (a - b)$.

Begin by substituting in the values for a and b.

$= -3 \cdot 4^2 - (-3 - 4)$

$= -3 \cdot 4^2 - -7$

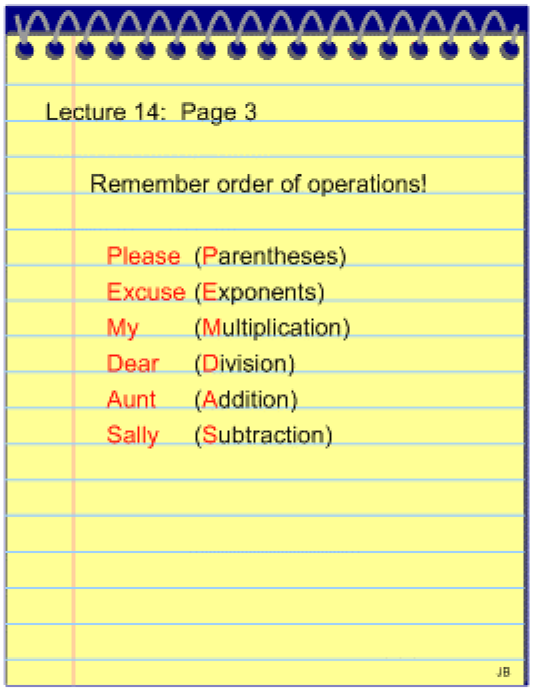
$= -3 \cdot 16 - -7$

$= -48 + 7$

$= -41$

JB

CEMP014-03



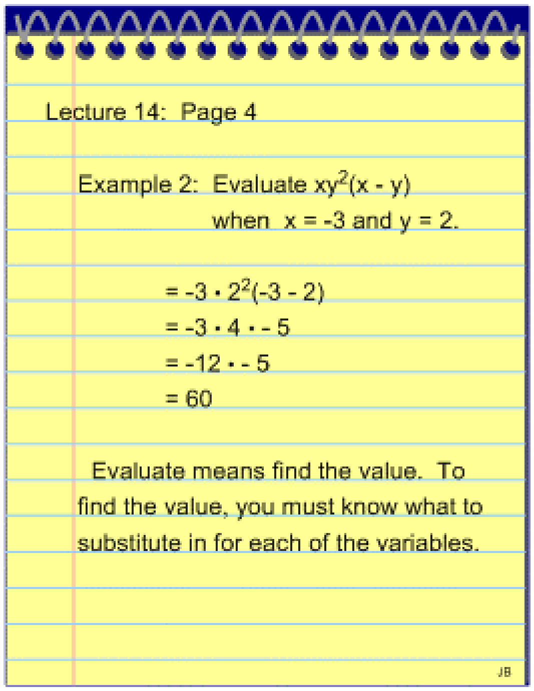
Lecture 14: Page 3

Remember order of operations!

Please	(Parentheses)
Excuse	(Exponents)
My	(Multiplication)
Dear	(Division)
Aunt	(Addition)
Sally	(Subtraction)

JB

CEMP014-04



Lecture 14: Page 4

Example 2: Evaluate $xy^2(x - y)$ when $x = -3$ and $y = 2$.

$= -3 \cdot 2^2(-3 - 2)$

$= -3 \cdot 4 \cdot - 5$

$= -12 \cdot - 5$

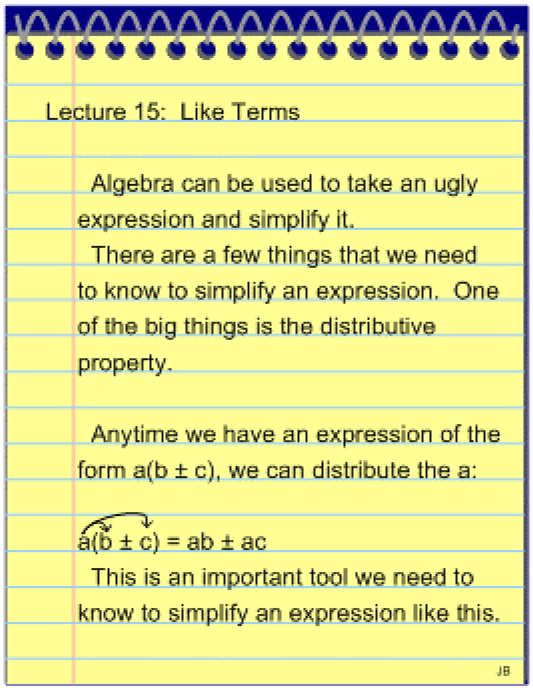
$= 60$

Evaluate means find the value. To find the value, you must know what to substitute in for each of the variables.

JB

Lecture 15 Notes

CEMP015-01



Lecture 15: Like Terms

Algebra can be used to take an ugly expression and simplify it.

There are a few things that we need to know to simplify an expression. One of the big things is the distributive property.

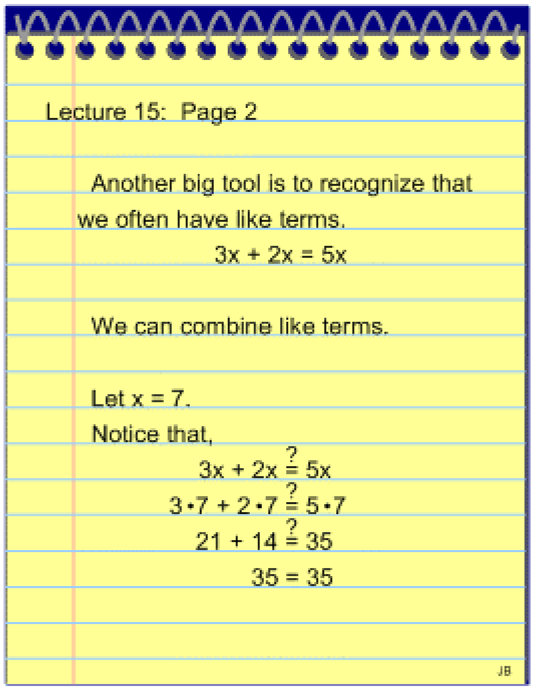
Anytime we have an expression of the form $a(b \pm c)$, we can distribute the a:

$a(b \pm c) = ab \pm ac$

This is an important tool we need to know to simplify an expression like this.

JB

CEMP015-02



Lecture 15: Page 2

Another big tool is to recognize that we often have like terms.

$$3x + 2x = 5x$$

We can combine like terms.

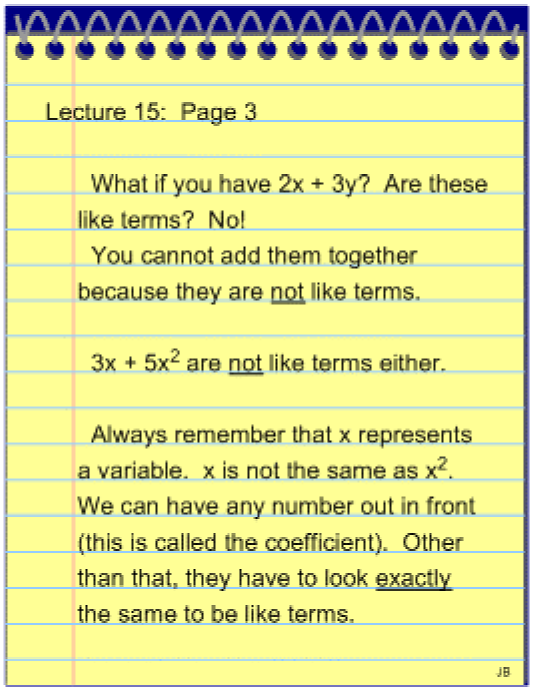
Let $x = 7$.

Notice that,

$$3x + 2x \overset{?}{=} 5x$$
$$3 \cdot 7 + 2 \cdot 7 \overset{?}{=} 5 \cdot 7$$
$$21 + 14 \overset{?}{=} 35$$
$$35 = 35$$

JB

CEMP015-03



Lecture 15: Page 3

What if you have $2x + 3y$? Are these like terms? No!

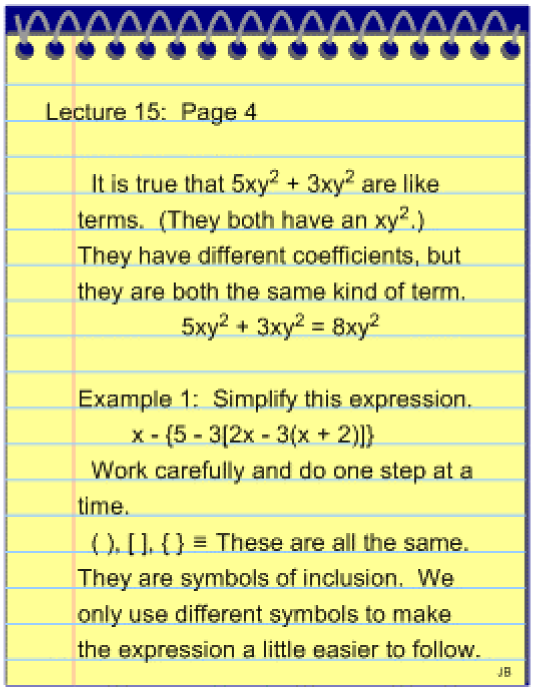
You cannot add them together because they are <u>not</u> like terms.

$3x + 5x^2$ are <u>not</u> like terms either.

Always remember that x represents a variable. x is not the same as x^2. We can have any number out in front (this is called the coefficient). Other than that, they have to look <u>exactly</u> the same to be like terms.

JB

CEMP015-04



Lecture 15: Page 4

It is true that $5xy^2 + 3xy^2$ are like terms. (They both have an xy^2.) They have different coefficients, but they are both the same kind of term.

$$5xy^2 + 3xy^2 = 8xy^2$$

Example 1: Simplify this expression.

$$x - \{5 - 3[2x - 3(x + 2)]\}$$

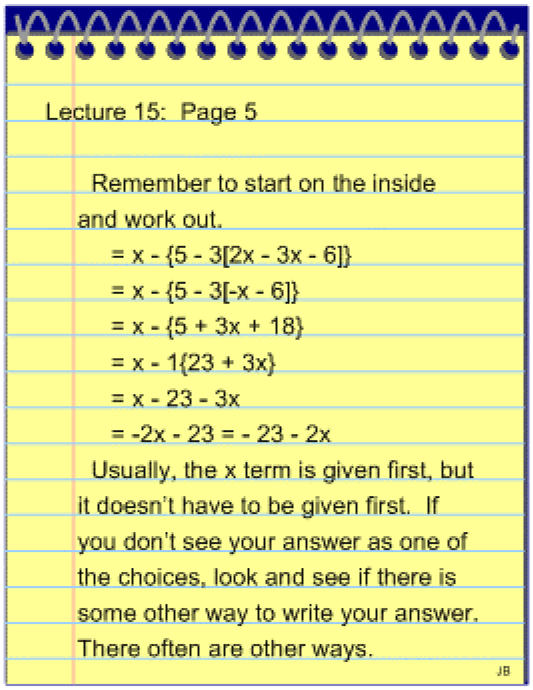
Work carefully and do one step at a time.

( ), [ ], { } ≡ These are all the same. They are symbols of inclusion. We only use different symbols to make the expression a little easier to follow.

JB

Lecture 15 Notes, Continued

CEMP015-05



Lecture 15: Page 5

Remember to start on the inside and work out.

$= x - \{5 - 3[2x - 3x - 6]\}$

$= x - \{5 - 3[-x - 6]\}$

$= x - \{5 + 3x + 18\}$

$= x - 1\{23 + 3x\}$

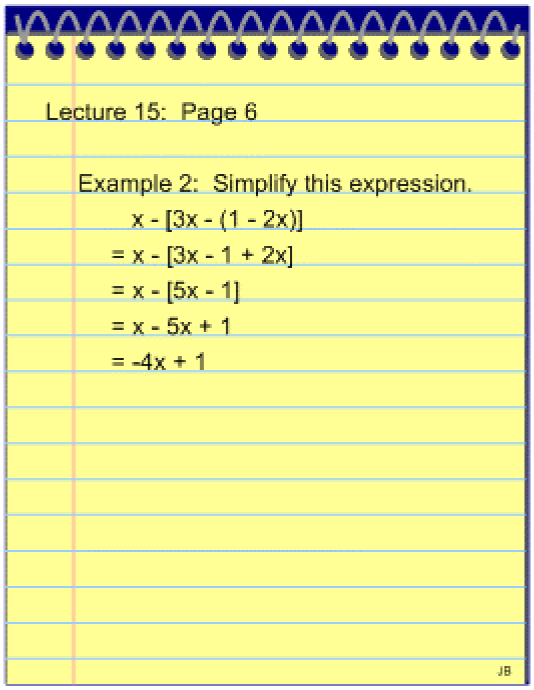
$= x - 23 - 3x$

$= -2x - 23 = - 23 - 2x$

Usually, the x term is given first, but it doesn't have to be given first. If you don't see your answer as one of the choices, look and see if there is some other way to write your answer. There often are other ways.

JB

CEMP015-06



Lecture 15: Page 6

Example 2: Simplify this expression.

$x - [3x - (1 - 2x)]$

$= x - [3x - 1 + 2x]$

$= x - [5x - 1]$

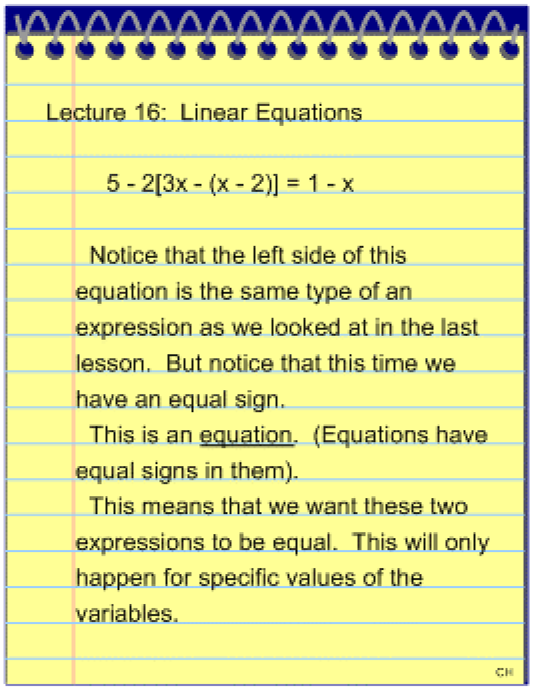
$= x - 5x + 1$

$= -4x + 1$

JB

Lecture 16 Notes

CEMP016-01



Lecture 16: Linear Equations

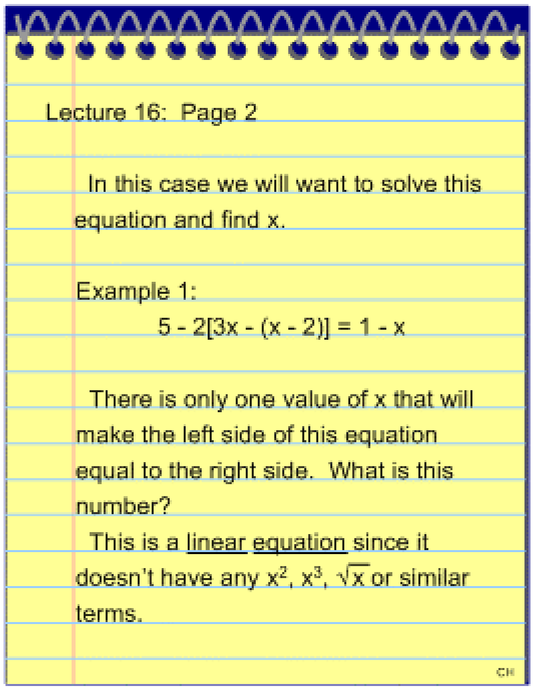
$$5 - 2[3x - (x - 2)] = 1 - x$$

Notice that the left side of this equation is the same type of an expression as we looked at in the last lesson. But notice that this time we have an equal sign.

This is an <u>equation</u>. (Equations have equal signs in them).

This means that we want these two expressions to be equal. This will only happen for specific values of the variables.

CEMP016-02



Lecture 16: Page 2

In this case we will want to solve this equation and find x.

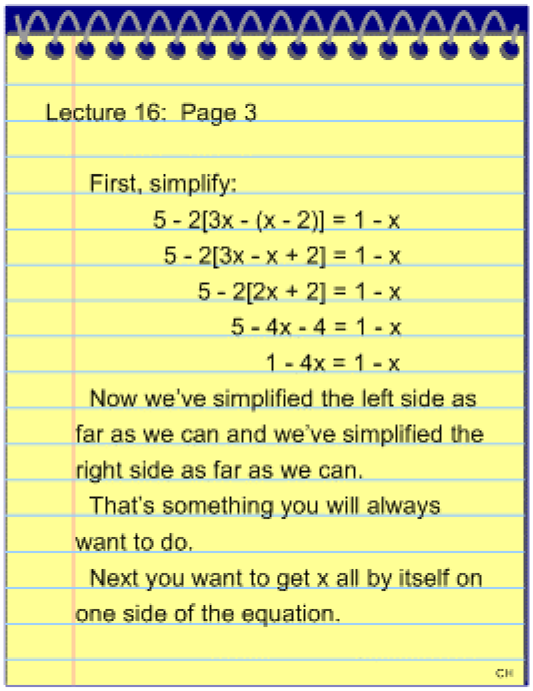
Example 1:

$$5 - 2[3x - (x - 2)] = 1 - x$$

There is only one value of x that will make the left side of this equation equal to the right side. What is this number?

This is a <u>linear</u> <u>equation</u> since it doesn't have any x^2, x^3, $\sqrt{x}$ or similar terms.

CEMP016-03



Lecture 16: Page 3

First, simplify:

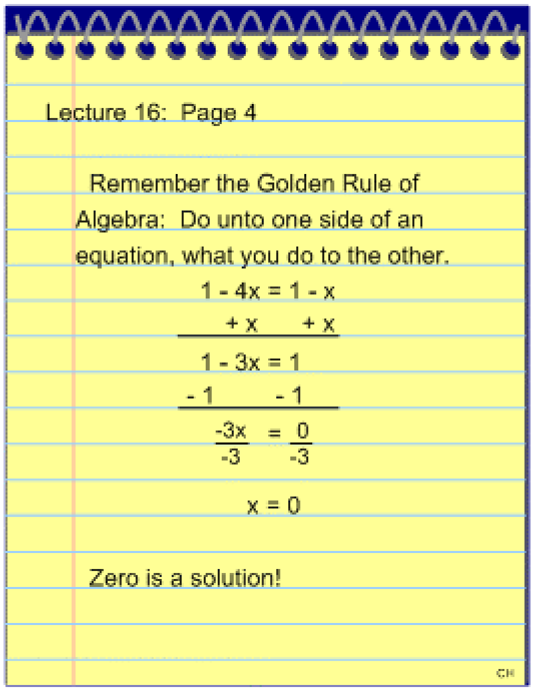
$$5 - 2[3x - (x - 2)] = 1 - x$$
$$5 - 2[3x - x + 2] = 1 - x$$
$$5 - 2[2x + 2] = 1 - x$$
$$5 - 4x - 4 = 1 - x$$
$$1 - 4x = 1 - x$$

Now we've simplified the left side as far as we can and we've simplified the right side as far as we can.

That's something you will always want to do.

Next you want to get x all by itself on one side of the equation.

CEMP016-04



Lecture 16: Page 4

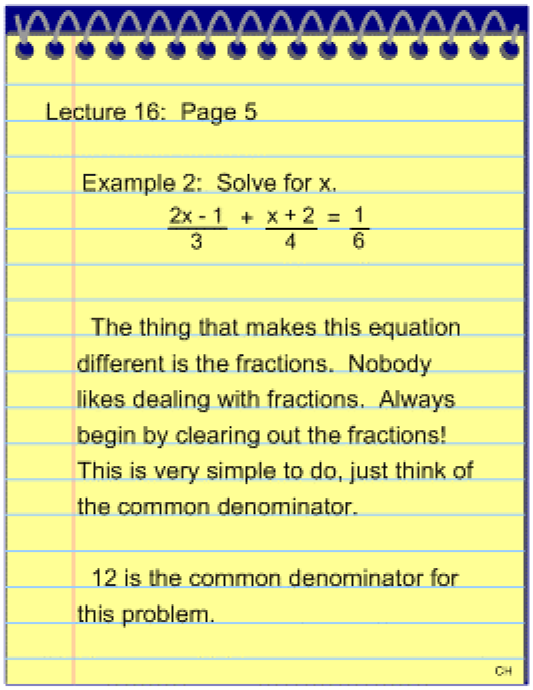
Remember the Golden Rule of Algebra: Do unto one side of an equation, what you do to the other.

$$1 - 4x = 1 - x$$
$$\underline{\quad + x \qquad + x}$$
$$1 - 3x = 1$$
$$\underline{- 1 \qquad - 1}$$
$$\frac{-3x}{-3} = \frac{0}{-3}$$
$$x = 0$$

Zero is a solution!

Lecture 16 Notes, Continued

CEMP016-05



Lecture 16: Page 5

Example 2: Solve for x.

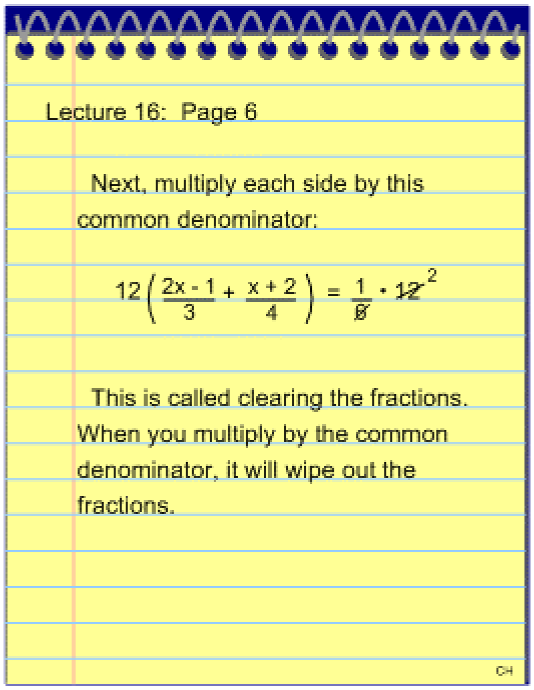
$$\frac{2x-1}{3} + \frac{x+2}{4} = \frac{1}{6}$$

The thing that makes this equation different is the fractions. Nobody likes dealing with fractions. Always begin by clearing out the fractions! This is very simple to do, just think of the common denominator.

12 is the common denominator for this problem.

CH

CEMP016-06



Lecture 16: Page 6

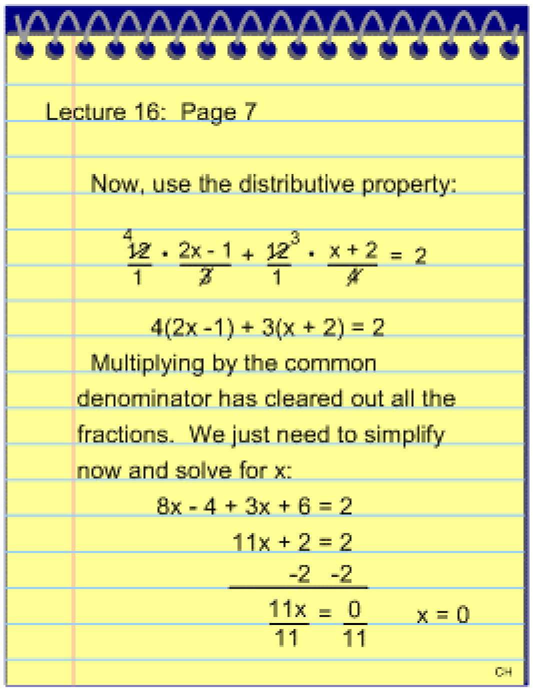
Next, multiply each side by this common denominator:

$$12\left(\frac{2x-1}{3} + \frac{x+2}{4}\right) = \frac{1}{\cancel{6}} \cdot \cancel{12}^{2}$$

This is called clearing the fractions. When you multiply by the common denominator, it will wipe out the fractions.

CH

CEMP016-07



Lecture 16: Page 7

Now, use the distributive property:

$$\frac{\cancel{12}^{4}}{1} \cdot \frac{2x-1}{\cancel{3}} + \frac{\cancel{12}^{3}}{1} \cdot \frac{x+2}{\cancel{4}} = 2$$

$$4(2x-1) + 3(x+2) = 2$$

Multiplying by the common denominator has cleared out all the fractions. We just need to simplify now and solve for x:

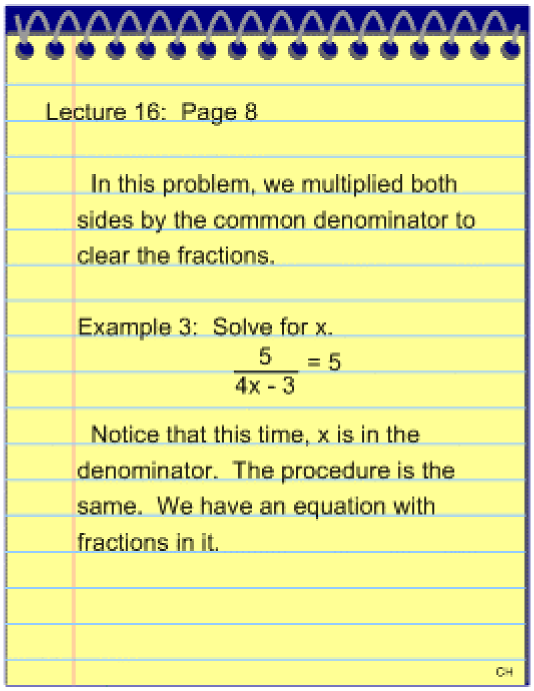
$$8x - 4 + 3x + 6 = 2$$

$$\begin{array}{r} 11x + 2 = 2 \\ -2 \quad -2 \\ \hline \end{array}$$

$$\frac{11x}{11} = \frac{0}{11} \qquad x = 0$$

CH

CEMP016-08



Lecture 16: Page 8

In this problem, we multiplied both sides by the common denominator to clear the fractions.

Example 3: Solve for x.

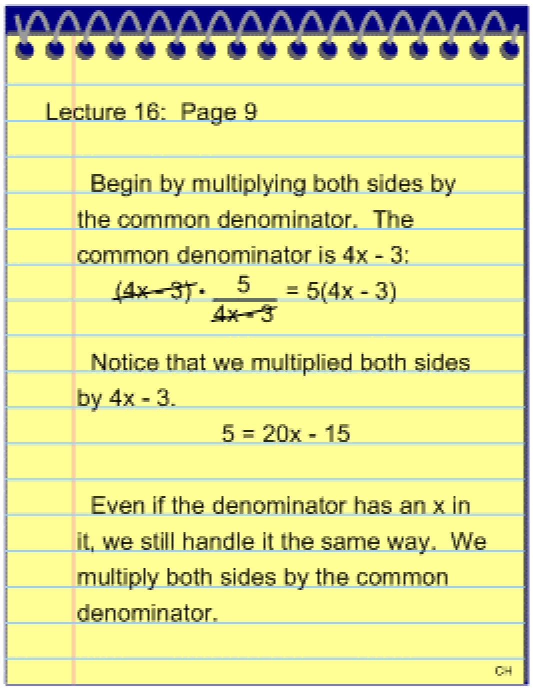
$$\frac{5}{4x-3} = 5$$

Notice that this time, x is in the denominator. The procedure is the same. We have an equation with fractions in it.

CH

Lecture 16 Notes, Continued

CEMP016-09



Lecture 16: Page 9

Begin by multiplying both sides by the common denominator. The common denominator is 4x - 3:

$$\cancel{(4x-3)} \cdot \frac{5}{\cancel{4x-3}} = 5(4x-3)$$

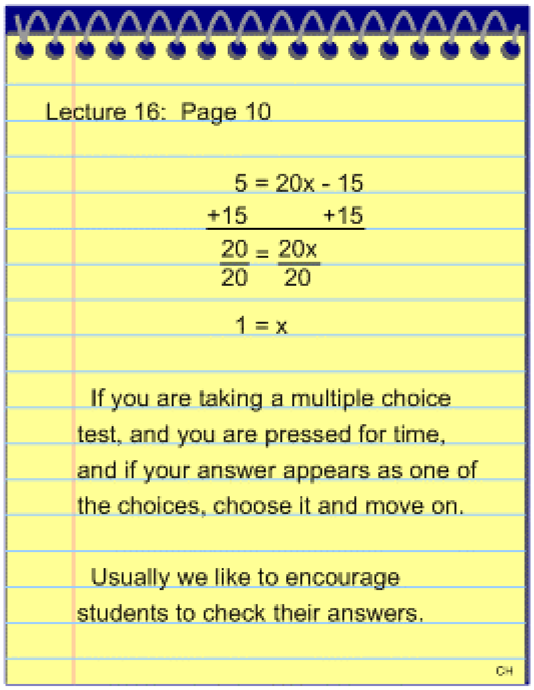
Notice that we multiplied both sides by 4x - 3.

$$5 = 20x - 15$$

Even if the denominator has an x in it, we still handle it the same way. We multiply both sides by the common denominator.

CH

CEMP016-10



Lecture 16: Page 10

$$\begin{array}{rcl} 5 & = & 20x - 15 \\ +15 & & +15 \\ \hline \frac{20}{20} & = & \frac{20x}{20} \end{array}$$

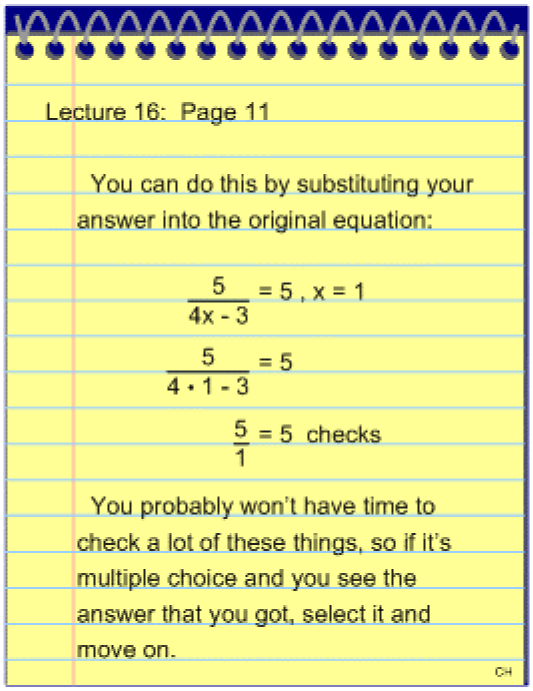
$$1 = x$$

If you are taking a multiple choice test, and you are pressed for time, and if your answer appears as one of the choices, choose it and move on.

Usually we like to encourage students to check their answers.

CH

CEMP016-11



Lecture 16: Page 11

You can do this by substituting your answer into the original equation:

$$\frac{5}{4x-3} = 5\text{ , } x = 1$$

$$\frac{5}{4 \cdot 1 - 3} = 5$$

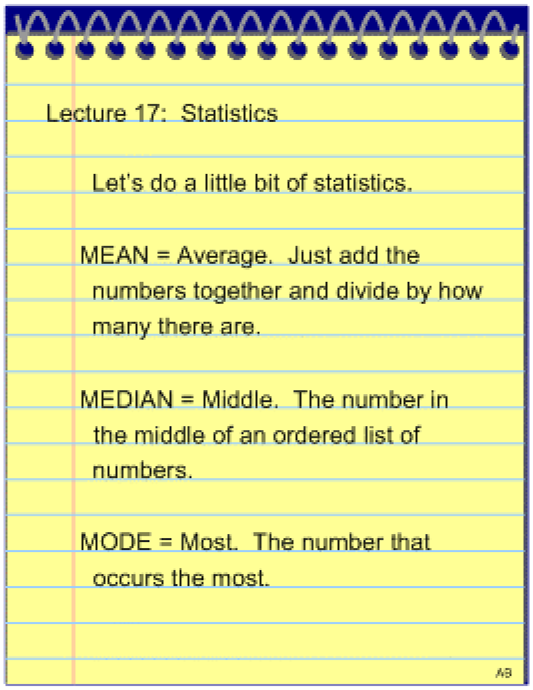
$$\frac{5}{1} = 5 \text{ checks}$$

You probably won't have time to check a lot of these things, so if it's multiple choice and you see the answer that you got, select it and move on.

CH

Lecture 17 Notes

CEMP017-01



Lecture 17: Statistics

Let's do a little bit of statistics.

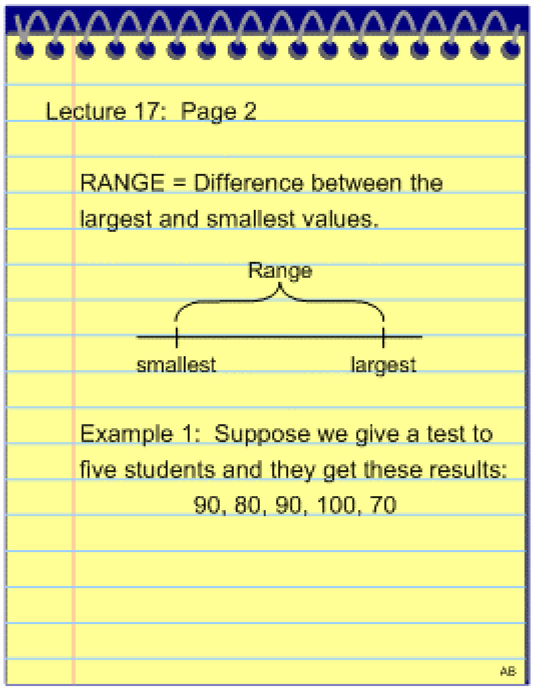
MEAN = Average. Just add the numbers together and divide by how many there are.

MEDIAN = Middle. The number in the middle of an ordered list of numbers.

MODE = Most. The number that occurs the most.

AB

CEMP017-02



Lecture 17: Page 2

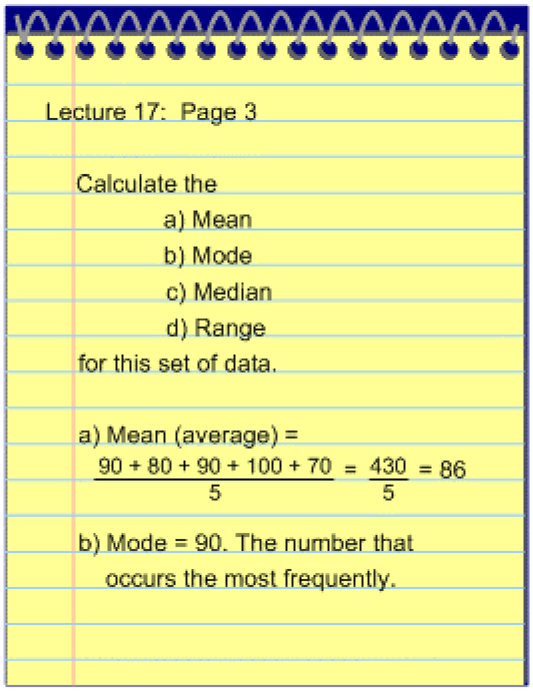
RANGE = Difference between the largest and smallest values.

Example 1: Suppose we give a test to five students and they get these results:

90, 80, 90, 100, 70

AB

CEMP017-03



Lecture 17: Page 3

Calculate the

a) Mean
b) Mode
c) Median
d) Range

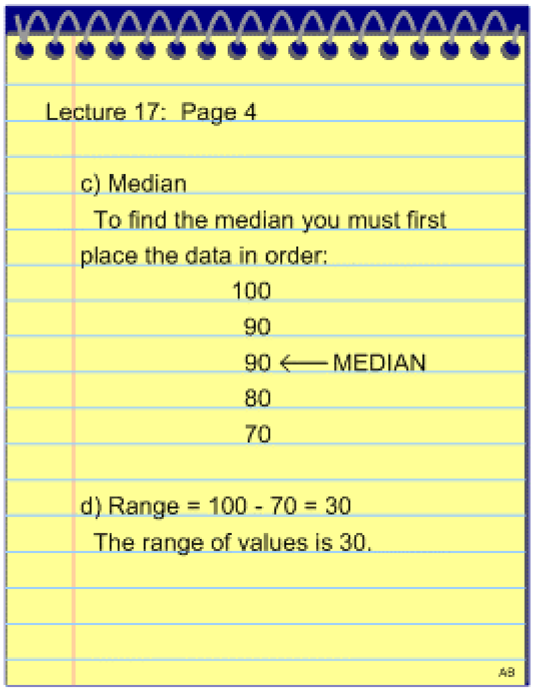
for this set of data.

a) Mean (average) =

$$\frac{90 + 80 + 90 + 100 + 70}{5} = \frac{430}{5} = 86$$

b) Mode = 90. The number that occurs the most frequently.

CEMP017-04



Lecture 17: Page 4

c) Median

To find the median you must first place the data in order:

100
90
90 ⟵ MEDIAN
80
70

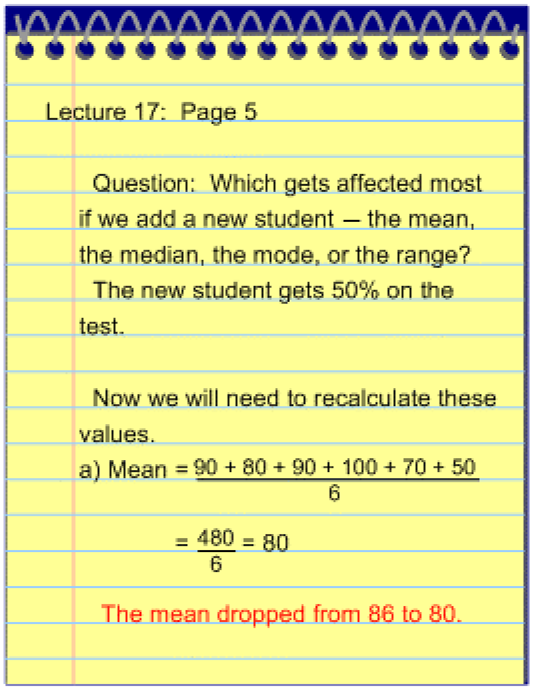
d) Range = 100 - 70 = 30

The range of values is 30.

AB

Lecture 17 Notes, Continued

CEMP017-05



Lecture 17: Page 5

Question: Which gets affected most if we add a new student – the mean, the median, the mode, or the range? The new student gets 50% on the test.

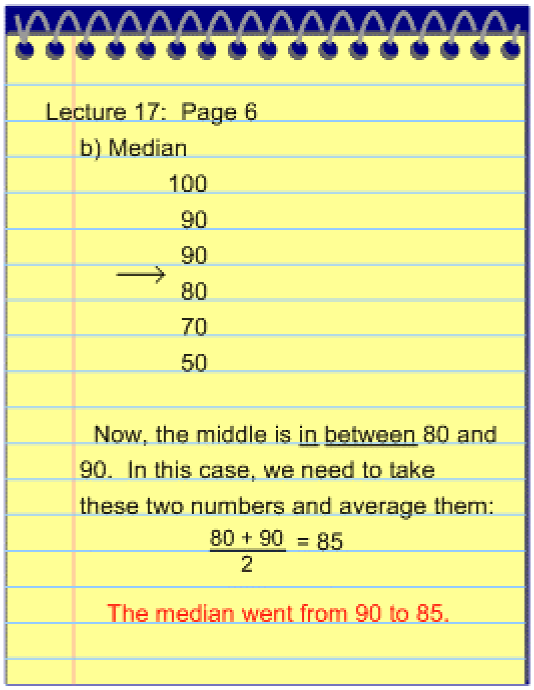
Now we will need to recalculate these values.

a) Mean $= \frac{90 + 80 + 90 + 100 + 70 + 50}{6}$

$= \frac{480}{6} = 80$

The mean dropped from 86 to 80.

CEMP017-06



Lecture 17: Page 6

b) Median

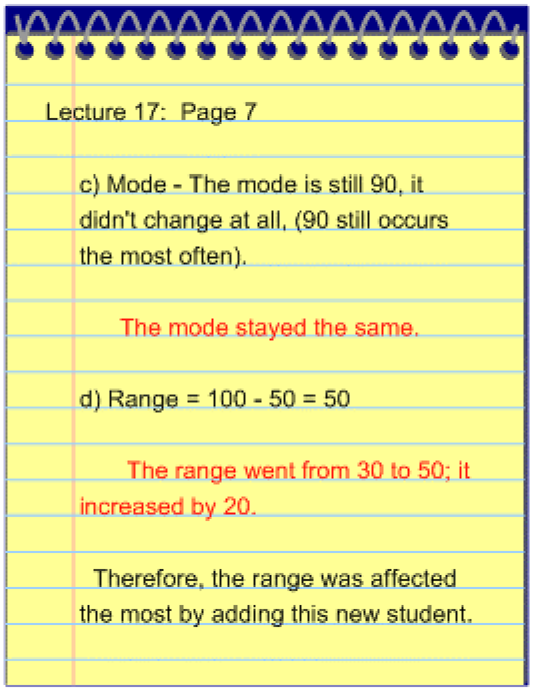
100
90
90
→
80
70
50

Now, the middle is in between 80 and 90. In this case, we need to take these two numbers and average them:

$\frac{80 + 90}{2} = 85$

The median went from 90 to 85.

CEMP017-07



Lecture 17: Page 7

c) Mode - The mode is still 90, it didn't change at all, (90 still occurs the most often).

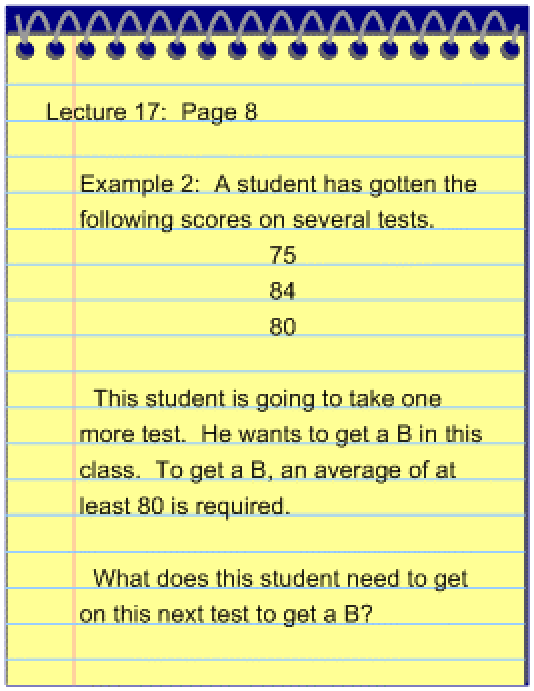
The mode stayed the same.

d) Range = 100 - 50 = 50

The range went from 30 to 50; it increased by 20.

Therefore, the range was affected the most by adding this new student.

CEMP017-08



Lecture 17: Page 8

Example 2: A student has gotten the following scores on several tests.

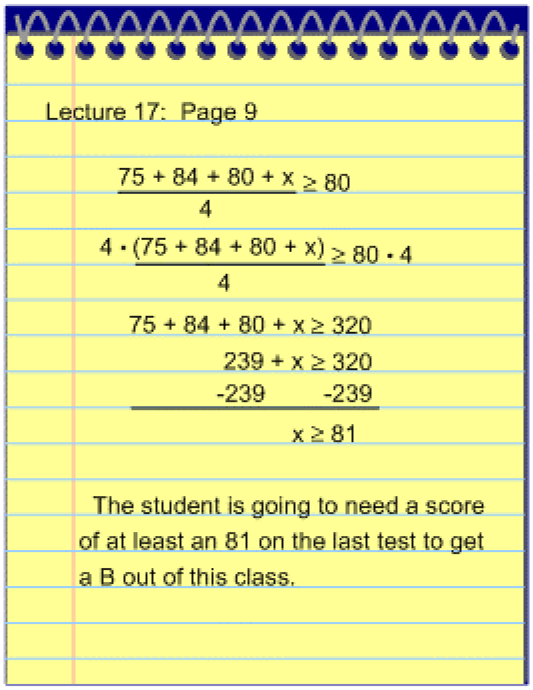
75
84
80

This student is going to take one more test. He wants to get a B in this class. To get a B, an average of at least 80 is required.

What does this student need to get on this next test to get a B?

Lecture 17 Notes, Continued

CEMP017-09



Lecture 17: Page 9

$$\frac{75 + 84 + 80 + x}{4} \geq 80$$

$$4 \cdot \frac{(75 + 84 + 80 + x)}{4} \geq 80 \cdot 4$$

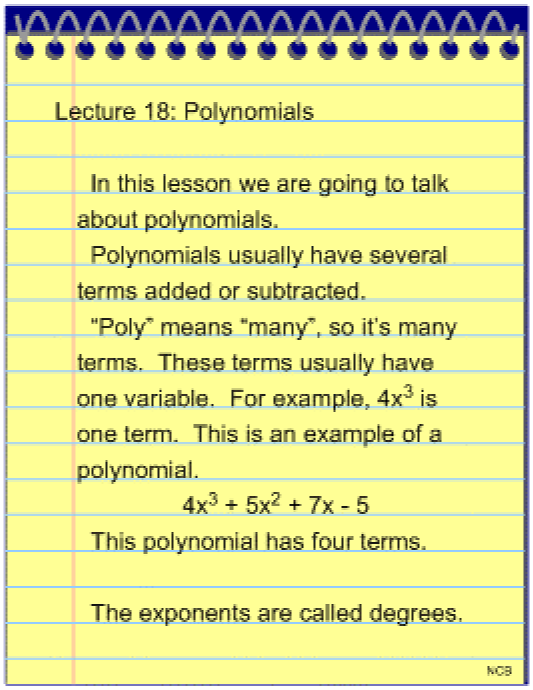
$$75 + 84 + 80 + x \geq 320$$

$$\begin{array}{rcl} 239 + x & \geq & 320 \\ \underline{-239} & & \underline{-239} \\ x & \geq & 81 \end{array}$$

The student is going to need a score of at least an 81 on the last test to get a B out of this class.

Lecture 18 Notes

CEMP018-01



Lecture 18: Polynomials

In this lesson we are going to talk about polynomials.

Polynomials usually have several terms added or subtracted.

"Poly" means "many", so it's many terms. These terms usually have one variable. For example, $4x^3$ is one term. This is an example of a polynomial.

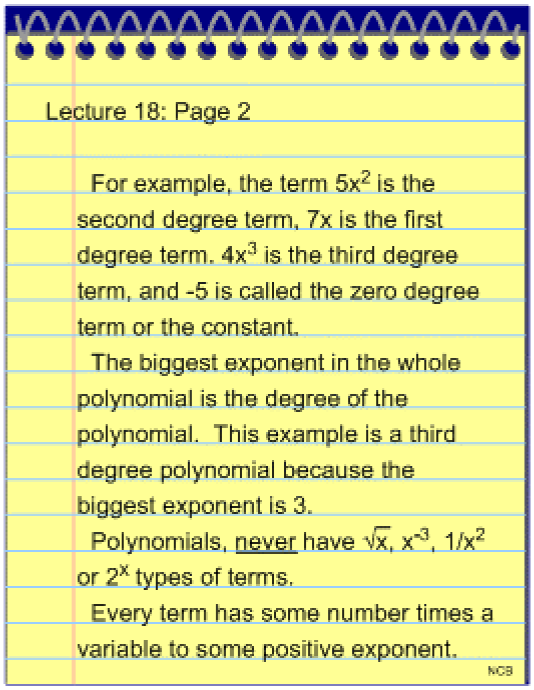
$$4x^3 + 5x^2 + 7x - 5$$

This polynomial has four terms.

The exponents are called degrees.

NCB

CEMP018-02



Lecture 18: Page 2

For example, the term $5x^2$ is the second degree term, $7x$ is the first degree term. $4x^3$ is the third degree term, and -5 is called the zero degree term or the constant.

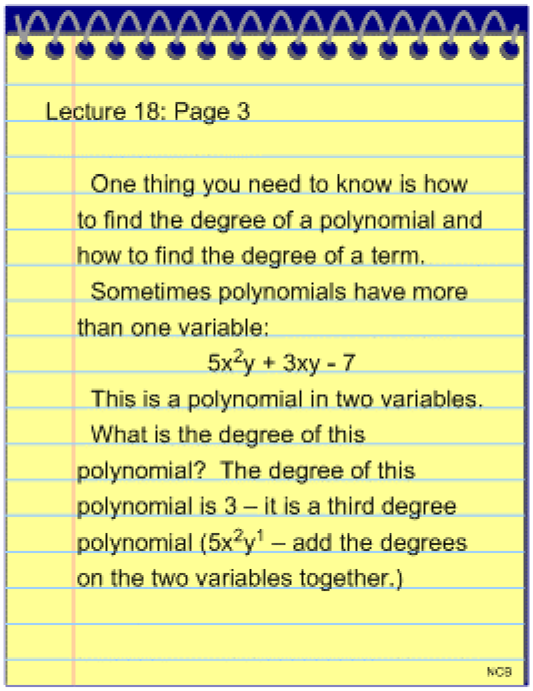
The biggest exponent in the whole polynomial is the degree of the polynomial. This example is a third degree polynomial because the biggest exponent is 3.

Polynomials, never have $\sqrt{x}$, x^{-3}, $1/x^2$ or 2^X types of terms.

Every term has some number times a variable to some positive exponent.

NCB

CEMP018-03



Lecture 18: Page 3

One thing you need to know is how to find the degree of a polynomial and how to find the degree of a term.

Sometimes polynomials have more than one variable:

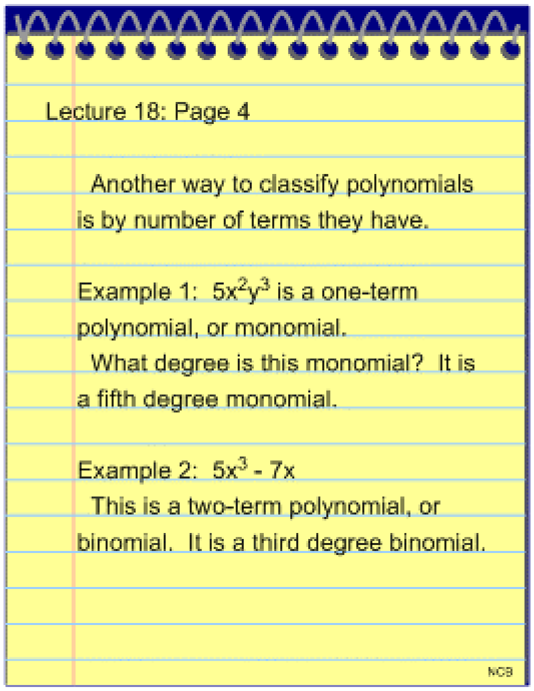
$$5x^2y + 3xy - 7$$

This is a polynomial in two variables.

What is the degree of this polynomial? The degree of this polynomial is 3 – it is a third degree polynomial ($5x^2y^1$ – add the degrees on the two variables together.)

NCB

CEMP018-04



Lecture 18: Page 4

Another way to classify polynomials is by number of terms they have.

Example 1: $5x^2y^3$ is a one-term polynomial, or monomial.

What degree is this monomial? It is a fifth degree monomial.

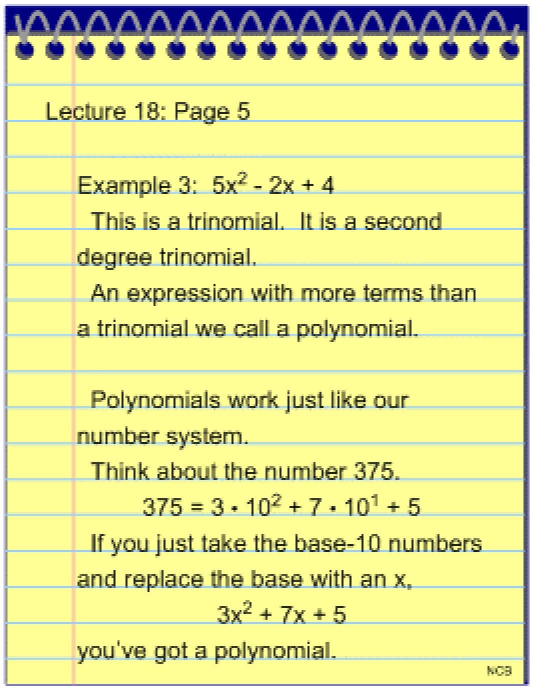
Example 2: $5x^3 - 7x$

This is a two-term polynomial, or binomial. It is a third degree binomial.

NCB

Lecture 18 Notes, Continued

CEMP018-05



Lecture 18: Page 5

Example 3: $5x^2 - 2x + 4$

This is a trinomial. It is a second degree trinomial.

An expression with more terms than a trinomial we call a polynomial.

Polynomials work just like our number system.

Think about the number 375.

$375 = 3 \cdot 10^2 + 7 \cdot 10^1 + 5$

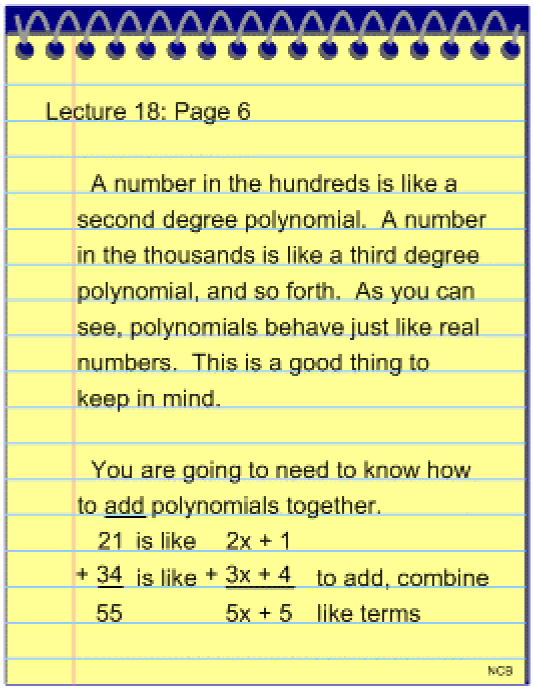
If you just take the base-10 numbers and replace the base with an x,

$3x^2 + 7x + 5$

you've got a polynomial.

NCB

CEMP018-06



Lecture 18: Page 6

A number in the hundreds is like a second degree polynomial. A number in the thousands is like a third degree polynomial, and so forth. As you can see, polynomials behave just like real numbers. This is a good thing to keep in mind.

You are going to need to know how to <u>add</u> polynomials together.

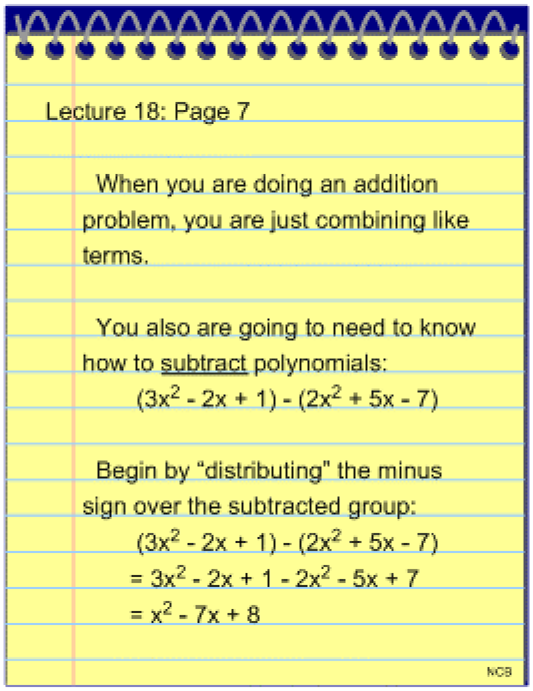
21 is like 2x + 1

+ $\underline{34}$ is like + $\underline{3x + 4}$ to add, combine

55 5x + 5 like terms

NCB

CEMP018-07



Lecture 18: Page 7

When you are doing an addition problem, you are just combining like terms.

You also are going to need to know how to <u>subtract</u> polynomials:

$(3x^2 - 2x + 1) - (2x^2 + 5x - 7)$

Begin by "distributing" the minus sign over the subtracted group:

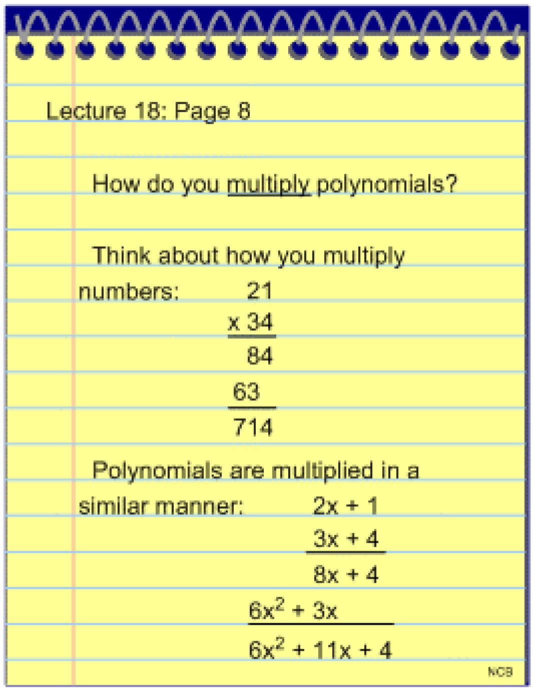
$(3x^2 - 2x + 1) - (2x^2 + 5x - 7)$

$= 3x^2 - 2x + 1 - 2x^2 - 5x + 7$

$= x^2 - 7x + 8$

NCB

CEMP018-08



Lecture 18: Page 8

How do you <u>multiply</u> polynomials?

Think about how you multiply numbers:

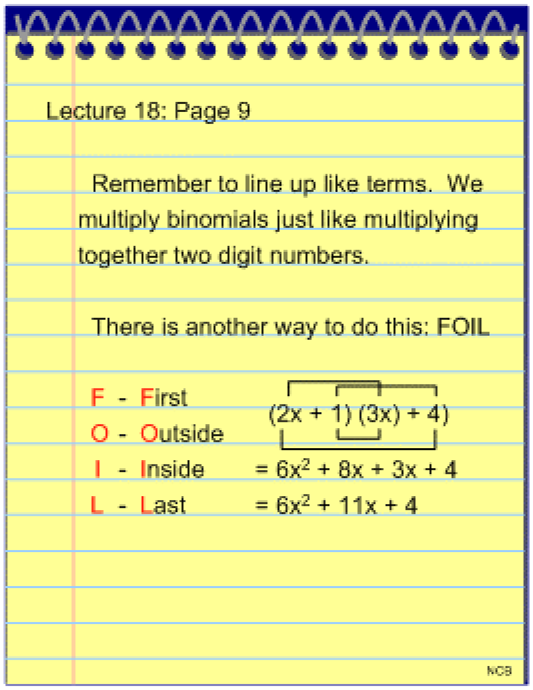
$$\begin{array}{r} 21 \\ \underline{\times\ 34} \\ 84 \\ \underline{63} \\ 714 \end{array}$$

Polynomials are multiplied in a similar manner:

$$\begin{array}{r} 2x + 1 \\ \underline{3x + 4} \\ 8x + 4 \\ \underline{6x^2 + 3x} \\ 6x^2 + 11x + 4 \end{array}$$

NCB

CEMP018-09



Lecture 18: Page 9

Remember to line up like terms. We multiply binomials just like multiplying together two digit numbers.

There is another way to do this: FOIL

F - First
O - Outside
I - Inside
L - Last

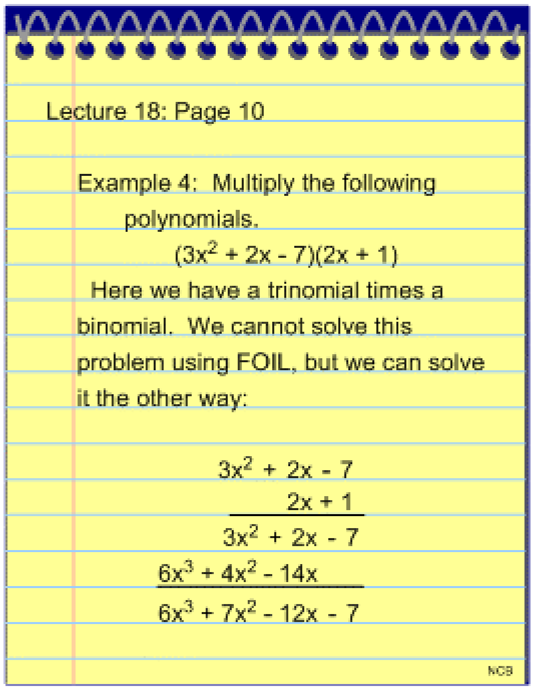
$(2x + 1) (3x) + 4)$

$= 6x^2 + 8x + 3x + 4$

$= 6x^2 + 11x + 4$

NCB

CEMP018-10



Lecture 18: Page 10

Example 4: Multiply the following polynomials.

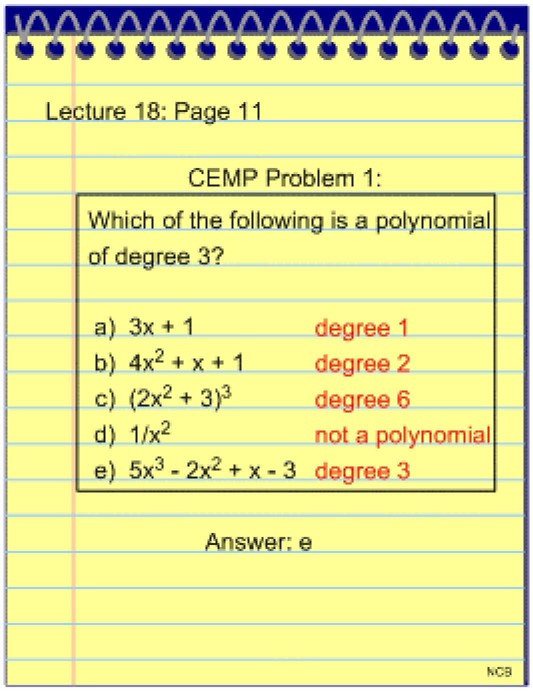
$(3x^2 + 2x - 7)(2x + 1)$

Here we have a trinomial times a binomial. We cannot solve this problem using FOIL, but we can solve it the other way:

$$\begin{array}{r} 3x^2 + 2x - 7 \\ \underline{2x + 1} \\ 3x^2 + 2x - 7 \\ \underline{6x^3 + 4x^2 - 14x} \\ 6x^3 + 7x^2 - 12x - 7 \end{array}$$

NCB

CEMP018-11



Lecture 18: Page 11

CEMP Problem 1:

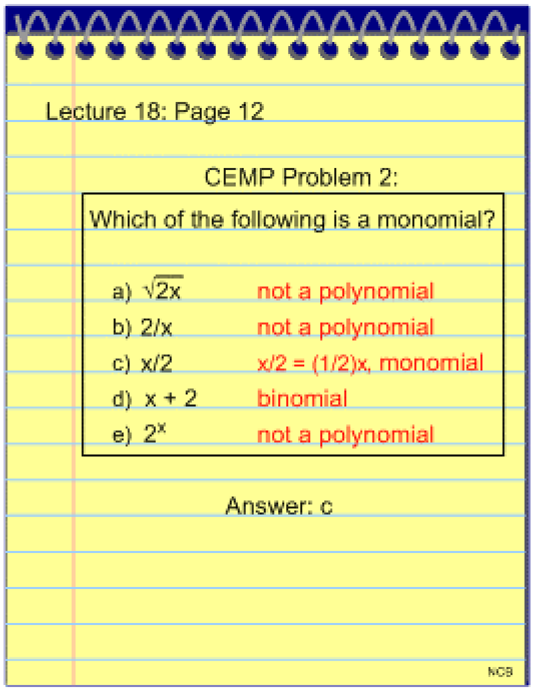
Which of the following is a polynomial of degree 3?

a) $3x + 1$	degree 1
b) $4x^2 + x + 1$	degree 2
c) $(2x^2 + 3)^3$	degree 6
d) $1/x^2$	not a polynomial
e) $5x^3 - 2x^2 + x - 3$	degree 3

Answer: e

NCB

CEMP018-12



Lecture 18: Page 12

CEMP Problem 2:

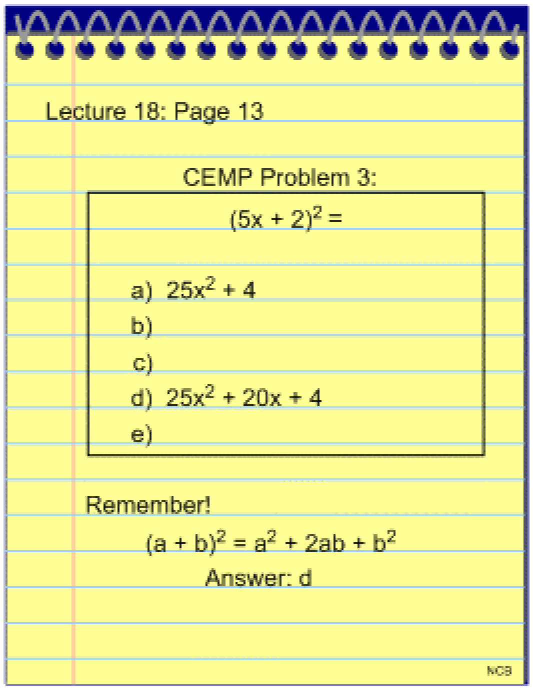
Which of the following is a monomial?

a) $\sqrt{2x}$	not a polynomial
b) $2/x$	not a polynomial
c) $x/2$	$x/2 = (1/2)x$, monomial
d) $x + 2$	binomial
e) 2^x	not a polynomial

Answer: c

NCB

CEMP018-13



Lecture 18: Page 13

CEMP Problem 3:

$(5x + 2)^2 =$

a) $25x^2 + 4$
b)
c)
d) $25x^2 + 20x + 4$
e)

Remember!

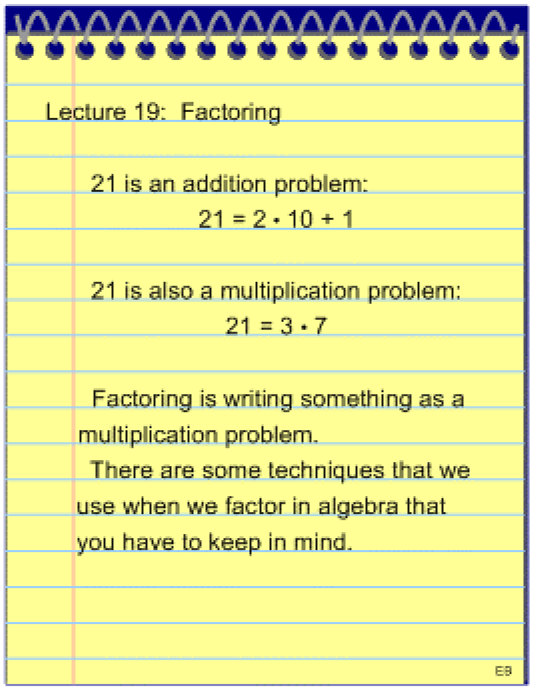
$(a + b)^2 = a^2 + 2ab + b^2$

Answer: d

NCB

Lecture 19 Notes

CEMP019-01



Lecture 19: Factoring

21 is an addition problem:

$$21 = 2 \cdot 10 + 1$$

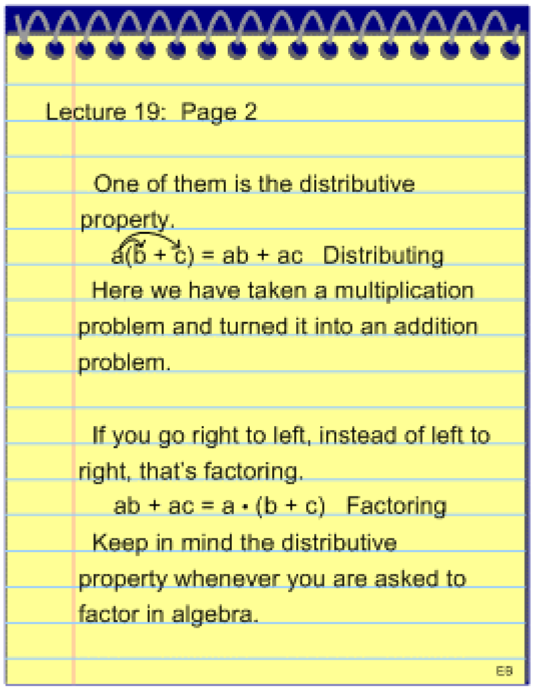
21 is also a multiplication problem:

$$21 = 3 \cdot 7$$

Factoring is writing something as a multiplication problem.

There are some techniques that we use when we factor in algebra that you have to keep in mind.

CEMP019-02



Lecture 19: Page 2

One of them is the distributive property.

$a(b + c) = ab + ac$ Distributing

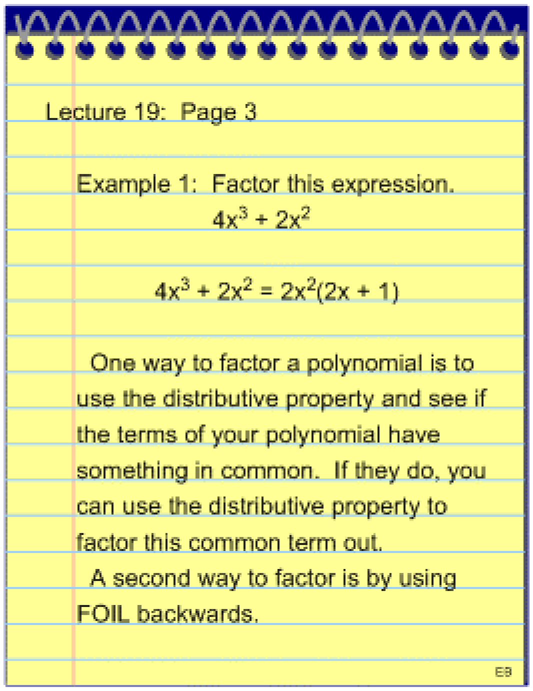
Here we have taken a multiplication problem and turned it into an addition problem.

If you go right to left, instead of left to right, that's factoring.

$ab + ac = a \cdot (b + c)$ Factoring

Keep in mind the distributive property whenever you are asked to factor in algebra.

CEMP019-03



Lecture 19: Page 3

Example 1: Factor this expression.

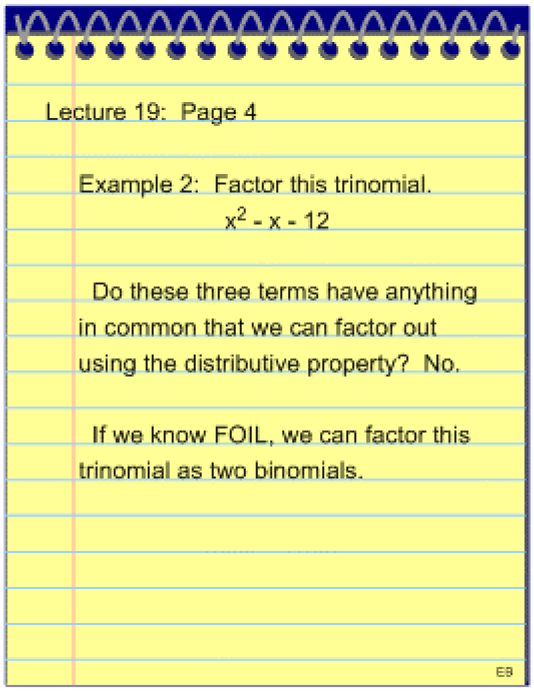
$$4x^3 + 2x^2$$

$$4x^3 + 2x^2 = 2x^2(2x + 1)$$

One way to factor a polynomial is to use the distributive property and see if the terms of your polynomial have something in common. If they do, you can use the distributive property to factor this common term out.

A second way to factor is by using FOIL backwards.

CEMP019-04



Lecture 19: Page 4

Example 2: Factor this trinomial.

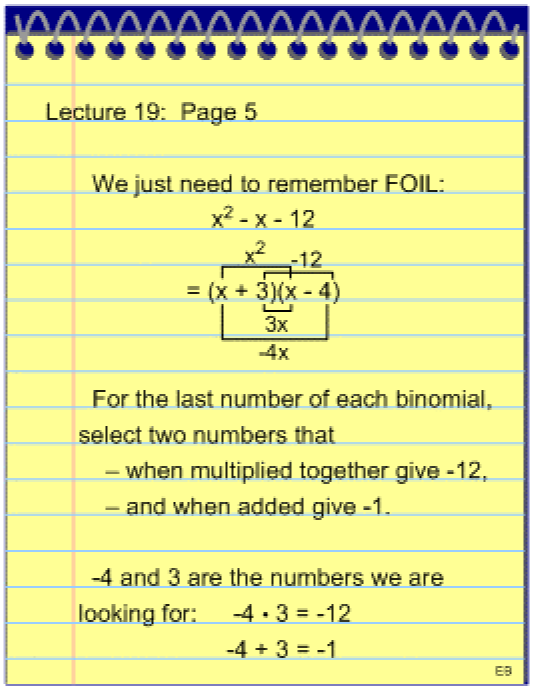
$$x^2 - x - 12$$

Do these three terms have anything in common that we can factor out using the distributive property? No.

If we know FOIL, we can factor this trinomial as two binomials.

Lecture 19 Notes, Continued

CEMP019-05



Lecture 19: Page 5

We just need to remember FOIL:

$x^2 - x - 12$

$$\overset{x^2 \qquad -12}{= (x + 3)(x - 4)}$$

$$\underset{-4x}{\underset{3x}{}}$$

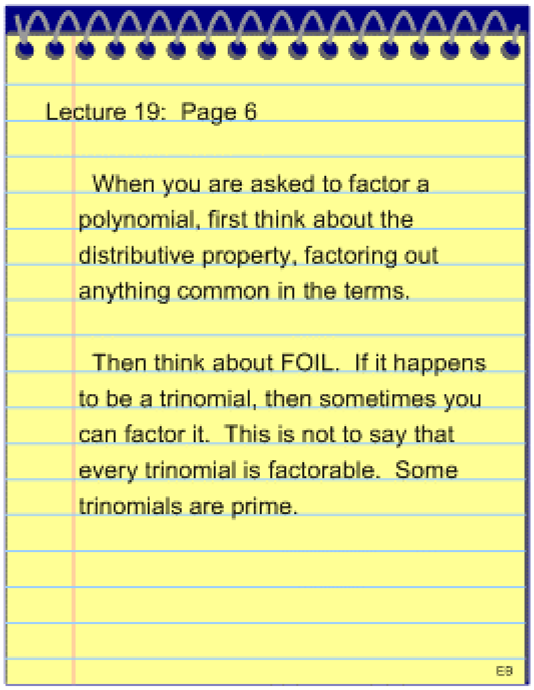
For the last number of each binomial, select two numbers that

- when multiplied together give -12,
- and when added give -1.

-4 and 3 are the numbers we are looking for: $-4 \cdot 3 = -12$

$-4 + 3 = -1$

CEMP019-06

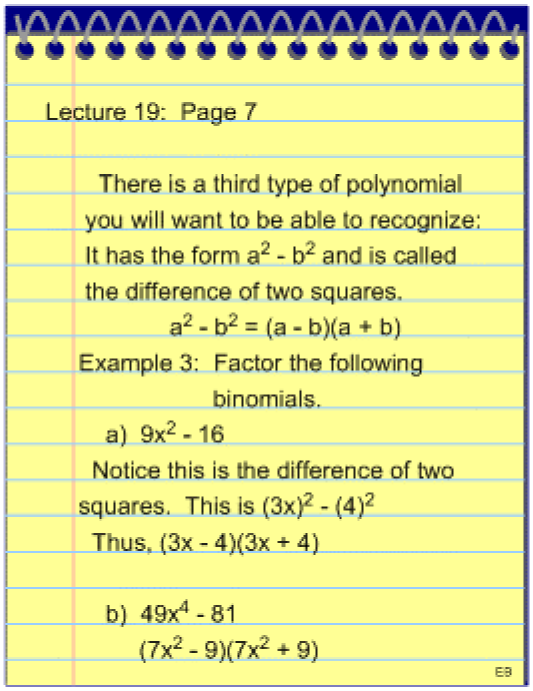


Lecture 19: Page 6

When you are asked to factor a polynomial, first think about the distributive property, factoring out anything common in the terms.

Then think about FOIL. If it happens to be a trinomial, then sometimes you can factor it. This is not to say that every trinomial is factorable. Some trinomials are prime.

CEMP019-07



Lecture 19: Page 7

There is a third type of polynomial you will want to be able to recognize: It has the form $a^2 - b^2$ and is called the difference of two squares.

$$a^2 - b^2 = (a - b)(a + b)$$

Example 3: Factor the following binomials.

a) $9x^2 - 16$

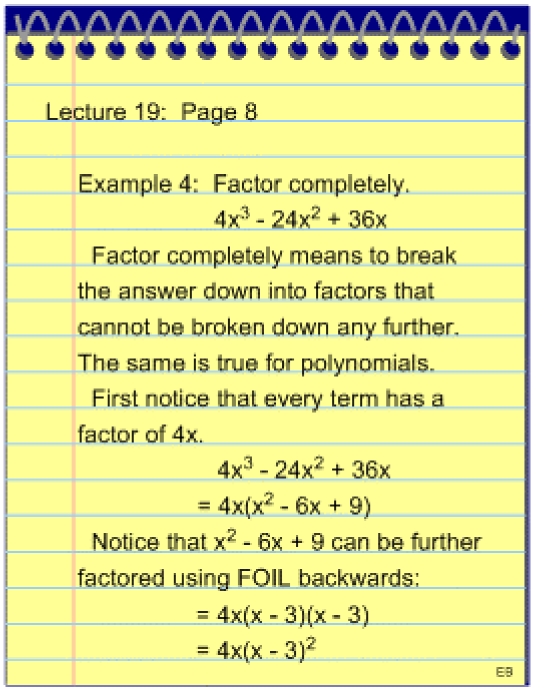
Notice this is the difference of two squares. This is $(3x)^2 - (4)^2$

Thus, $(3x - 4)(3x + 4)$

b) $49x^4 - 81$

$(7x^2 - 9)(7x^2 + 9)$

CEMP019-08



Lecture 19: Page 8

Example 4: Factor completely.

$4x^3 - 24x^2 + 36x$

Factor completely means to break the answer down into factors that cannot be broken down any further. The same is true for polynomials.

First notice that every term has a factor of 4x.

$4x^3 - 24x^2 + 36x$

$= 4x(x^2 - 6x + 9)$

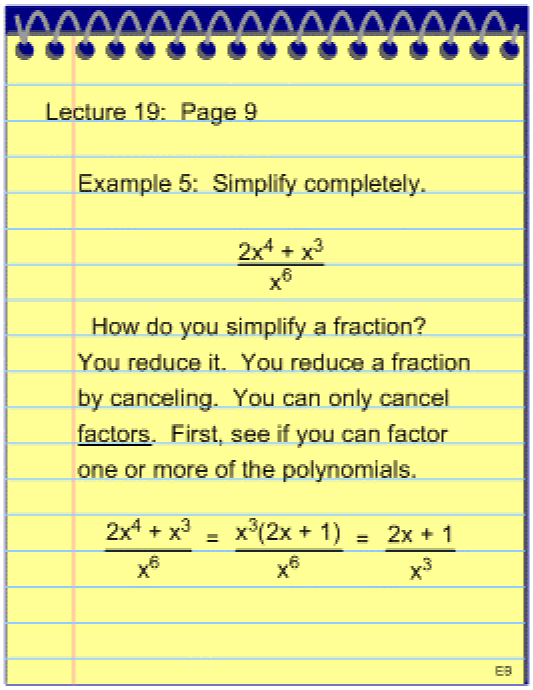
Notice that $x^2 - 6x + 9$ can be further factored using FOIL backwards:

$= 4x(x - 3)(x - 3)$

$= 4x(x - 3)^2$

Lecture 19 Notes, Continued

CEMP019-09



Lecture 19: Page 9

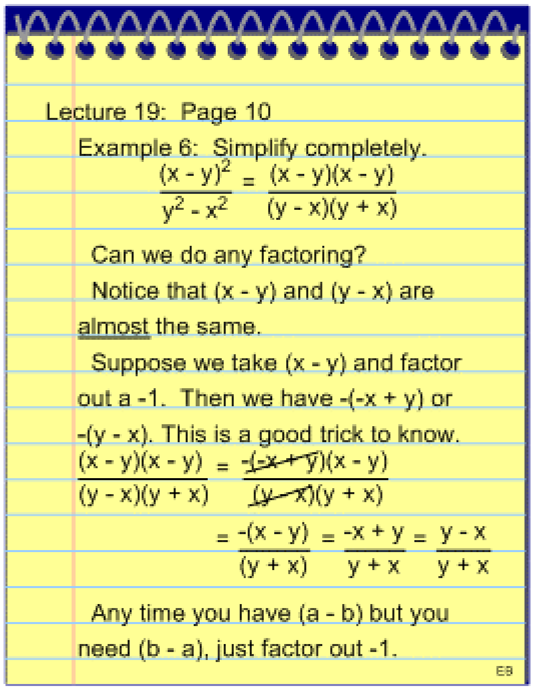
Example 5: Simplify completely.

$$\frac{2x^4 + x^3}{x^6}$$

How do you simplify a fraction? You reduce it. You reduce a fraction by canceling. You can only cancel factors. First, see if you can factor one or more of the polynomials.

$$\frac{2x^4 + x^3}{x^6} = \frac{x^3(2x + 1)}{x^6} = \frac{2x + 1}{x^3}$$

CEMP019-10



Lecture 19: Page 10

Example 6: Simplify completely.

$$\frac{(x - y)^2}{y^2 - x^2} = \frac{(x - y)(x - y)}{(y - x)(y + x)}$$

Can we do any factoring?

Notice that (x - y) and (y - x) are almost the same.

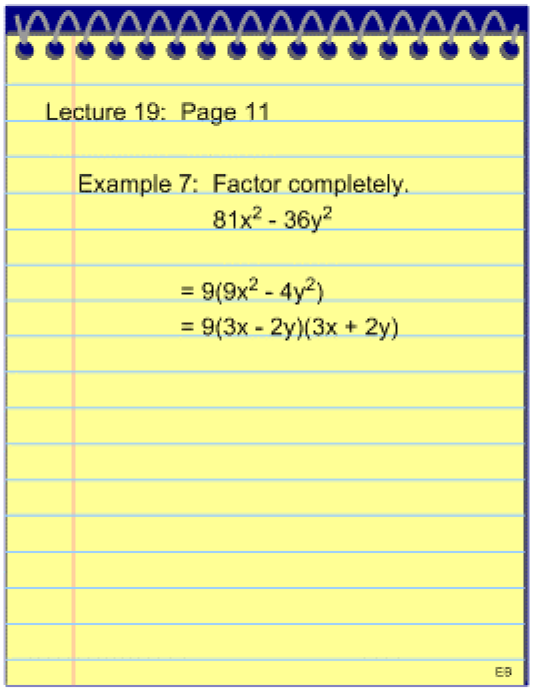
Suppose we take (x - y) and factor out a -1. Then we have -(-x + y) or -(y - x). This is a good trick to know.

$$\frac{(x - y)(x - y)}{(y - x)(y + x)} = \frac{-\cancel{(-x + y)}(x - y)}{\cancel{(y - x)}(y + x)}$$

$$= \frac{-(x - y)}{(y + x)} = \frac{-x + y}{y + x} = \frac{y - x}{y + x}$$

Any time you have (a - b) but you need (b - a), just factor out -1.

CEMP019-11



Lecture 19: Page 11

Example 7: Factor completely.

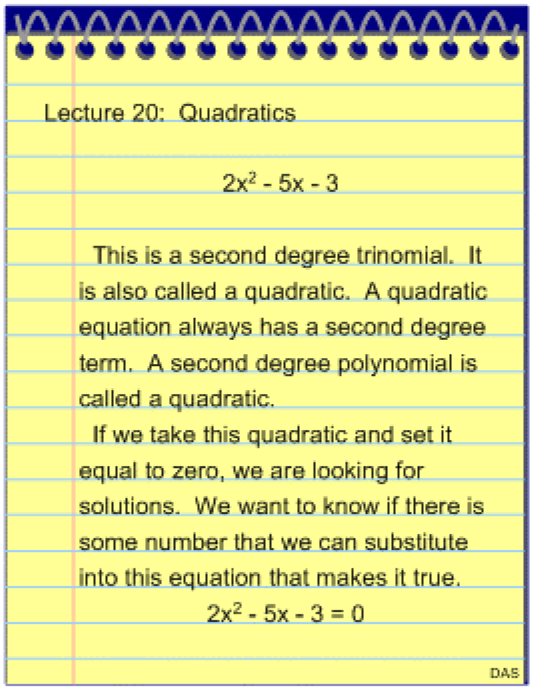
$$81x^2 - 36y^2$$

$$= 9(9x^2 - 4y^2)$$

$$= 9(3x - 2y)(3x + 2y)$$

Lecture 20 Notes

CEMP020-01



Lecture 20: Quadratics

$$2x^2 - 5x - 3$$

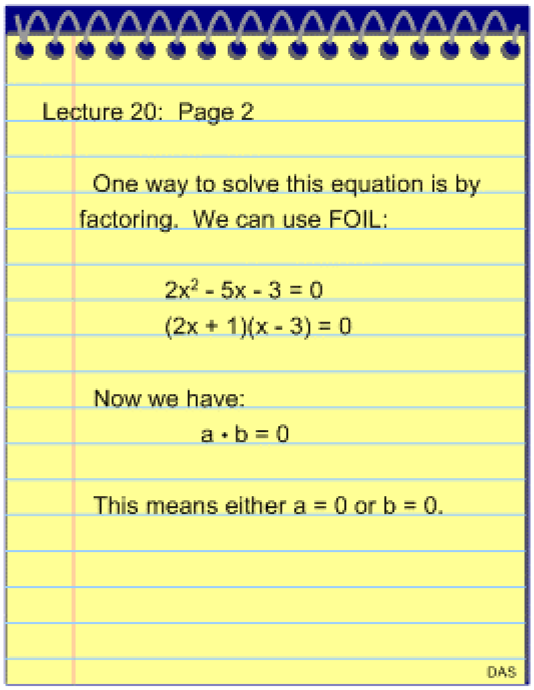
This is a second degree trinomial. It is also called a quadratic. A quadratic equation always has a second degree term. A second degree polynomial is called a quadratic.

If we take this quadratic and set it equal to zero, we are looking for solutions. We want to know if there is some number that we can substitute into this equation that makes it true.

$$2x^2 - 5x - 3 = 0$$

DAS

CEMP020-02



Lecture 20: Page 2

One way to solve this equation is by factoring. We can use FOIL:

$$2x^2 - 5x - 3 = 0$$
$$(2x + 1)(x - 3) = 0$$

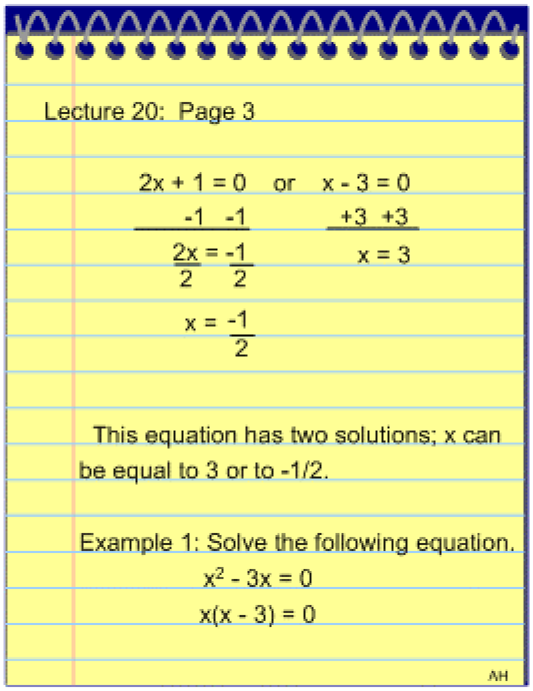
Now we have:

$$a \cdot b = 0$$

This means either a = 0 or b = 0.

DAS

CEMP020-03



Lecture 20: Page 3

$$2x + 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$\underline{\;\;-1 \;\; -1} \qquad \underline{+3 \;\; +3}$$
$$\frac{2x}{2} = \frac{-1}{2} \qquad x = 3$$
$$x = \frac{-1}{2}$$

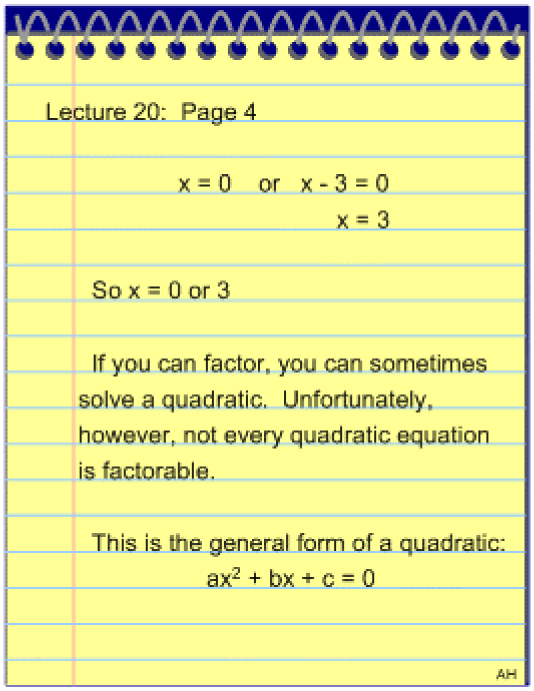
This equation has two solutions; x can be equal to 3 or to -1/2.

Example 1: Solve the following equation.

$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$

AH

CEMP020-04



Lecture 20: Page 4

$$x = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 3$$

So x = 0 or 3

If you can factor, you can sometimes solve a quadratic. Unfortunately, however, not every quadratic equation is factorable.

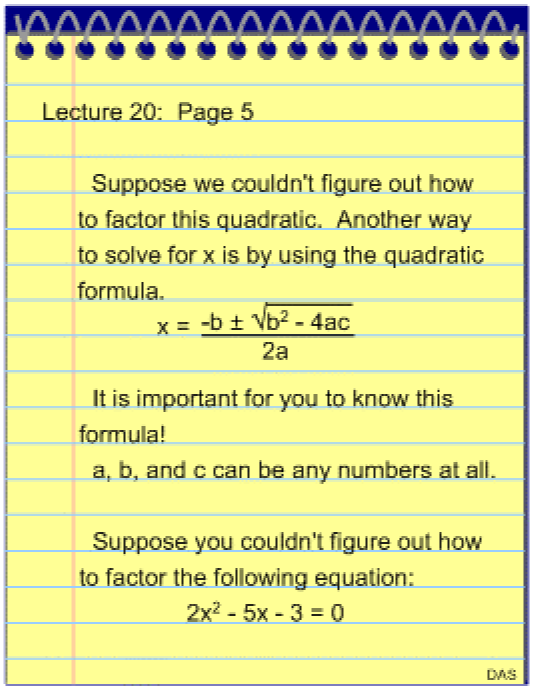
This is the general form of a quadratic:

$$ax^2 + bx + c = 0$$

AH

Lecture 20 Notes, Continued

CEMP020-05



Lecture 20: Page 5

Suppose we couldn't figure out how to factor this quadratic. Another way to solve for x is by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is important for you to know this formula!

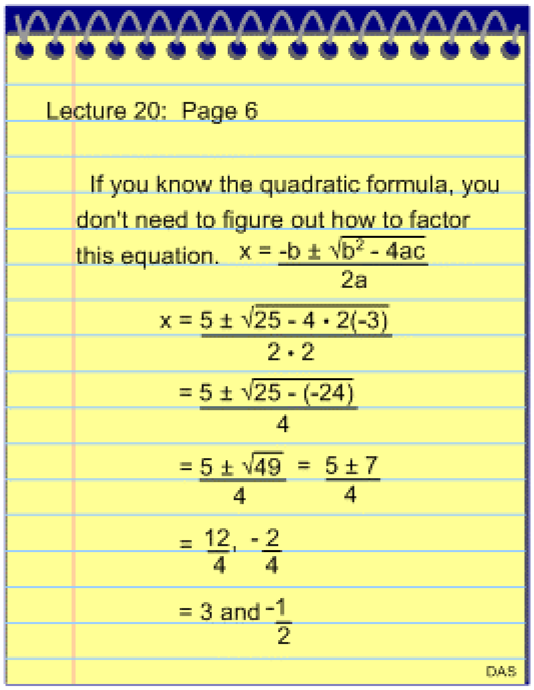
a, b, and c can be any numbers at all.

Suppose you couldn't figure out how to factor the following equation:

$$2x^2 - 5x - 3 = 0$$

DAS

CEMP020-06



Lecture 20: Page 6

If you know the quadratic formula, you don't need to figure out how to factor this equation. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{5 \pm \sqrt{25 - 4 \cdot 2(-3)}}{2 \cdot 2}$$

$$= \frac{5 \pm \sqrt{25 - (-24)}}{4}$$

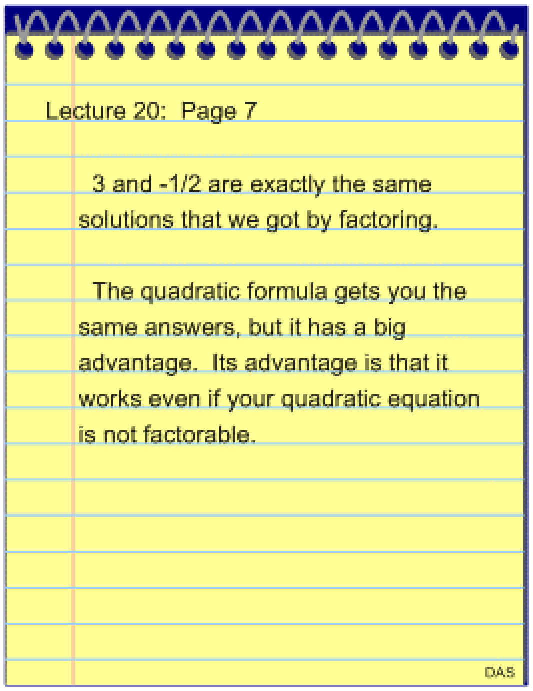
$$= \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4}$$

$$= \frac{12}{4}, -\frac{2}{4}$$

$$= 3 \text{ and } -\frac{1}{2}$$

DAS

CEMP020-07



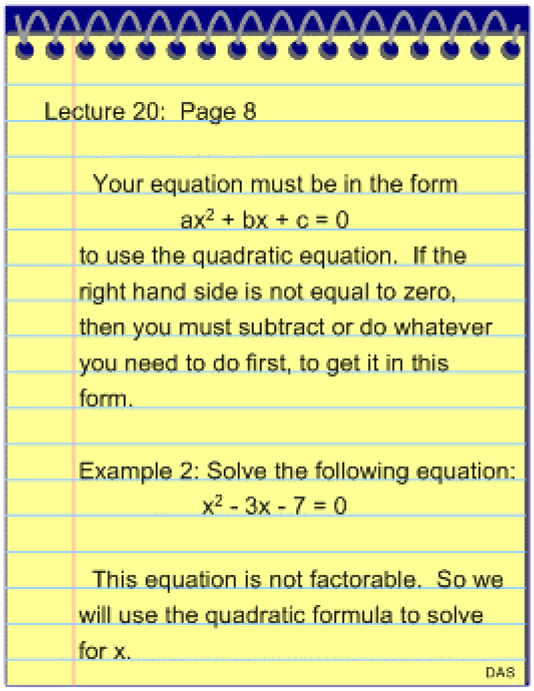
Lecture 20: Page 7

3 and -1/2 are exactly the same solutions that we got by factoring.

The quadratic formula gets you the same answers, but it has a big advantage. Its advantage is that it works even if your quadratic equation is not factorable.

DAS

CEMP020-08



Lecture 20: Page 8

Your equation must be in the form

$$ax^2 + bx + c = 0$$

to use the quadratic equation. If the right hand side is not equal to zero, then you must subtract or do whatever you need to do first, to get it in this form.

Example 2: Solve the following equation:

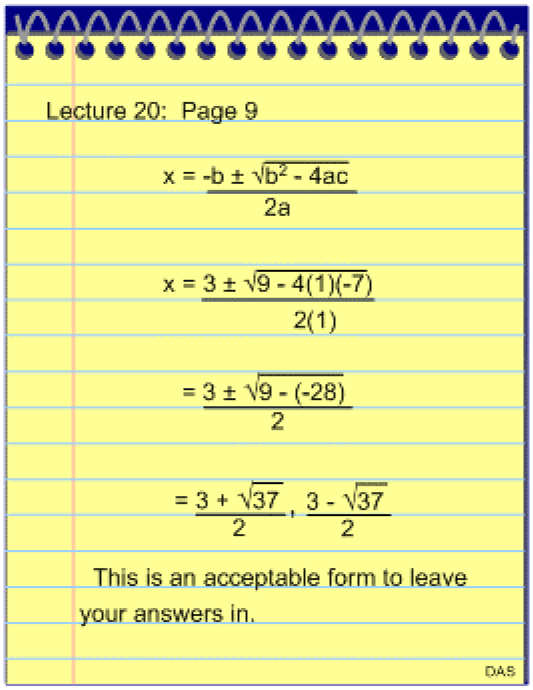
$$x^2 - 3x - 7 = 0$$

This equation is not factorable. So we will use the quadratic formula to solve for x.

DAS

Lecture 20 Notes, Continued

CEMP020-09



Lecture 20: Page 9

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{9 - 4(1)(-7)}}{2(1)}$$

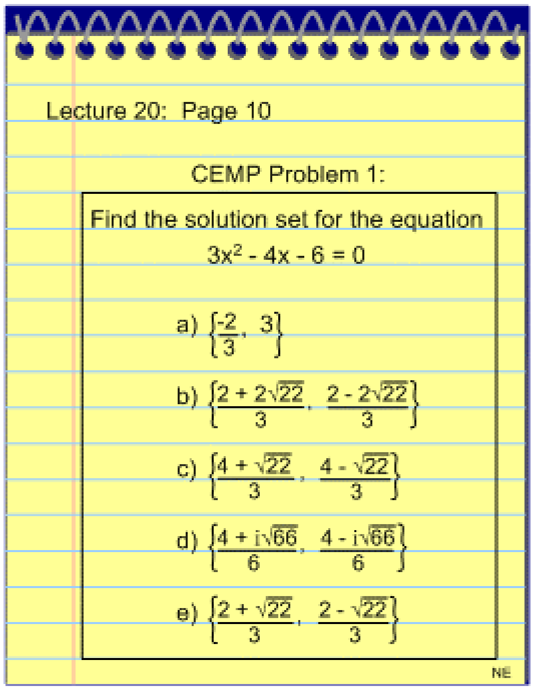
$$= \frac{3 \pm \sqrt{9 - (-28)}}{2}$$

$$= \frac{3 + \sqrt{37}}{2}, \frac{3 - \sqrt{37}}{2}$$

This is an acceptable form to leave your answers in.

DAS

CEMP020-10



Lecture 20: Page 10

CEMP Problem 1:

Find the solution set for the equation

$$3x^2 - 4x - 6 = 0$$

a) $\left\{\frac{-2}{3}, 3\right\}$

b) $\left\{\frac{2 + 2\sqrt{22}}{3}, \frac{2 - 2\sqrt{22}}{3}\right\}$

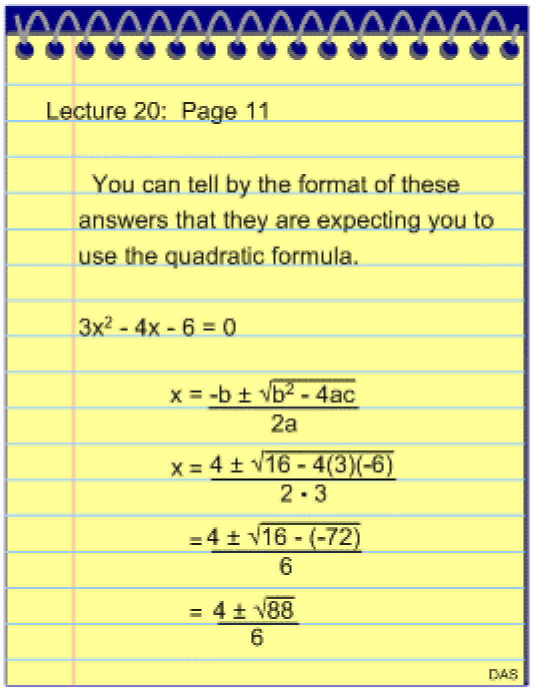
c) $\left\{\frac{4 + \sqrt{22}}{3}, \frac{4 - \sqrt{22}}{3}\right\}$

d) $\left\{\frac{4 + i\sqrt{66}}{6}, \frac{4 - i\sqrt{66}}{6}\right\}$

e) $\left\{\frac{2 + \sqrt{22}}{3}, \frac{2 - \sqrt{22}}{3}\right\}$

NE

CEMP020-11



Lecture 20: Page 11

You can tell by the format of these answers that they are expecting you to use the quadratic formula.

$3x^2 - 4x - 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

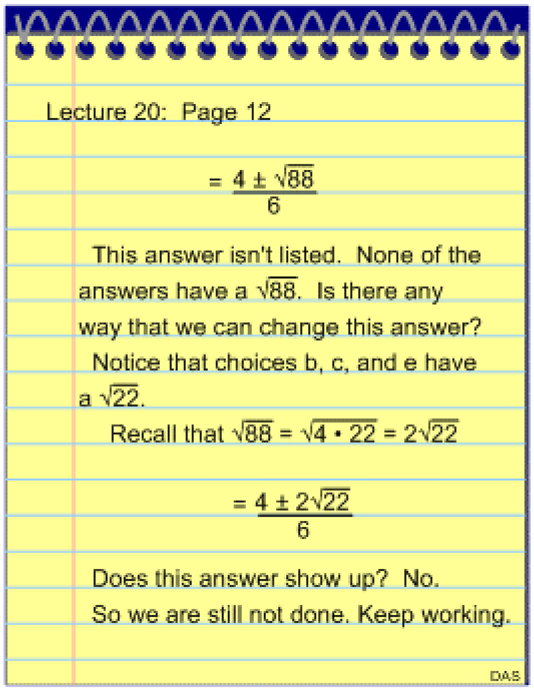
$$x = \frac{4 \pm \sqrt{16 - 4(3)(-6)}}{2 \cdot 3}$$

$$= \frac{4 \pm \sqrt{16 - (-72)}}{6}$$

$$= \frac{4 \pm \sqrt{88}}{6}$$

DAS

CEMP020-12



Lecture 20: Page 12

$$= \frac{4 \pm \sqrt{88}}{6}$$

This answer isn't listed. None of the answers have a $\sqrt{88}$. Is there any way that we can change this answer?

Notice that choices b, c, and e have a $\sqrt{22}$.

Recall that $\sqrt{88} = \sqrt{4 \cdot 22} = 2\sqrt{22}$

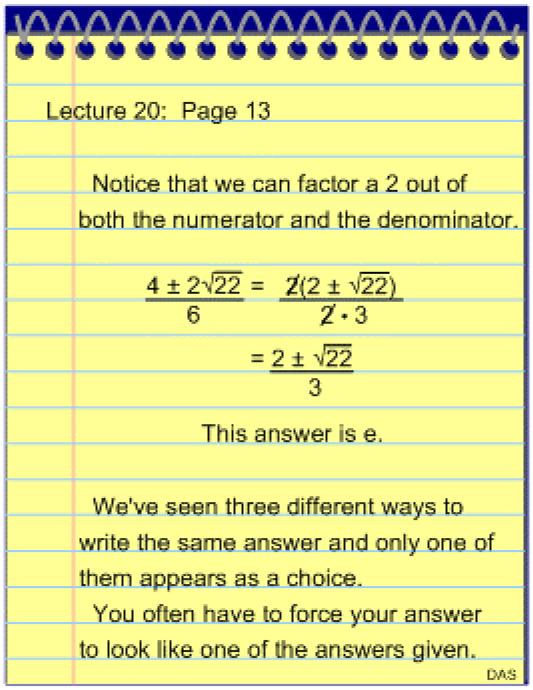
$$= \frac{4 \pm 2\sqrt{22}}{6}$$

Does this answer show up? No.

So we are still not done. Keep working.

DAS

CEMP020-13



Lecture 20: Page 13

Notice that we can factor a 2 out of both the numerator and the denominator.

$$\frac{4 \pm 2\sqrt{22}}{6} = \frac{\cancel{2}(2 \pm \sqrt{22})}{\cancel{2} \cdot 3}$$

$$= \frac{2 \pm \sqrt{22}}{3}$$

This answer is e.

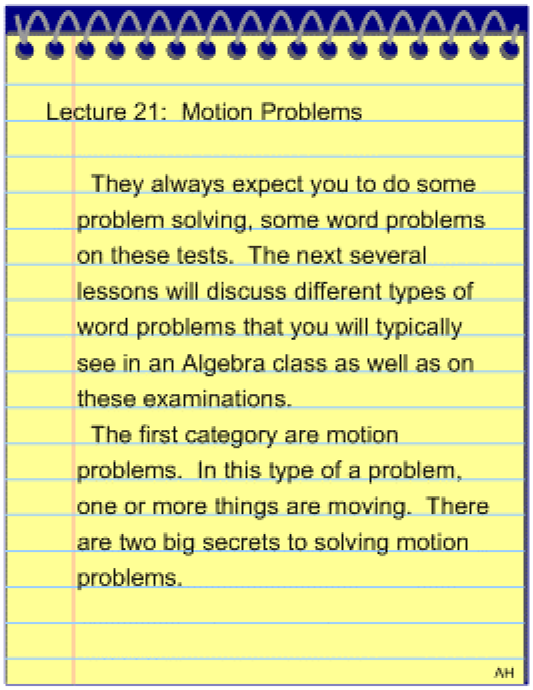
We've seen three different ways to write the same answer and only one of them appears as a choice.

You often have to force your answer to look like one of the answers given.

DAS

Lecture 21 Notes

CEMP021-01



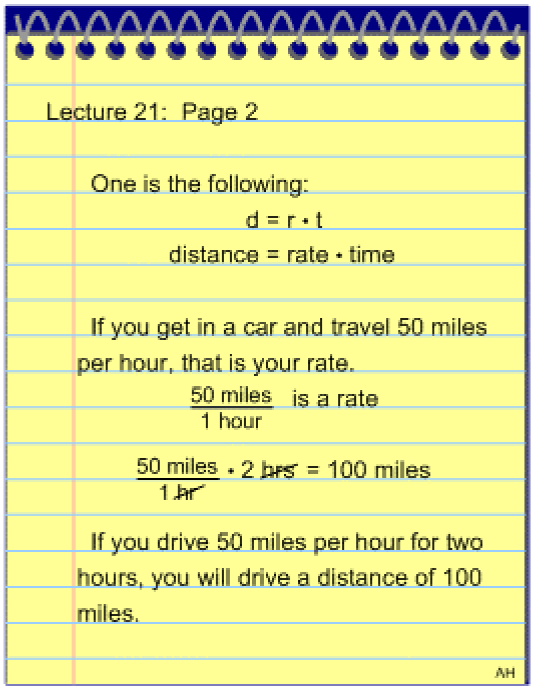
Lecture 21: Motion Problems

They always expect you to do some problem solving, some word problems on these tests. The next several lessons will discuss different types of word problems that you will typically see in an Algebra class as well as on these examinations.

The first category are motion problems. In this type of a problem, one or more things are moving. There are two big secrets to solving motion problems.

AH

CEMP021-02



Lecture 21: Page 2

One is the following:

$$d = r \cdot t$$

$$\text{distance} = \text{rate} \cdot \text{time}$$

If you get in a car and travel 50 miles per hour, that is your rate.

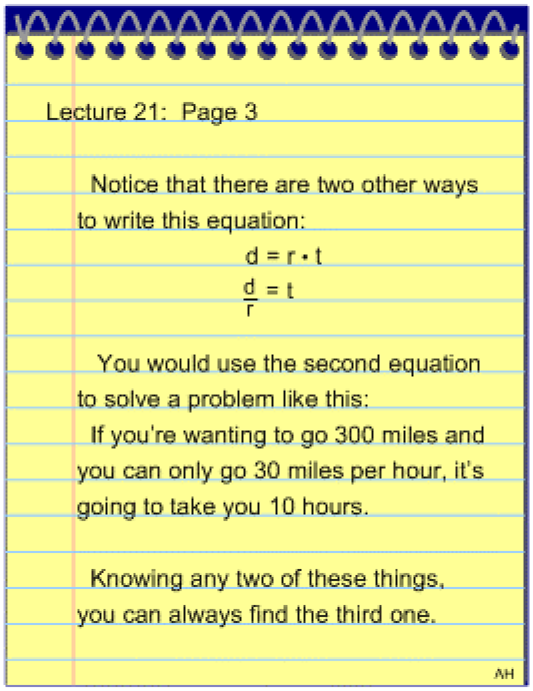
$$\frac{50\text{ miles}}{1\text{ hour}} \quad \text{is a rate}$$

$$\frac{50\text{ miles}}{1\,\cancel{\text{hr}}} \cdot 2\,\cancel{\text{hrs}} = 100\text{ miles}$$

If you drive 50 miles per hour for two hours, you will drive a distance of 100 miles.

AH

CEMP021-03



Lecture 21: Page 3

Notice that there are two other ways to write this equation:

$$d = r \cdot t$$

$$\frac{d}{r} = t$$

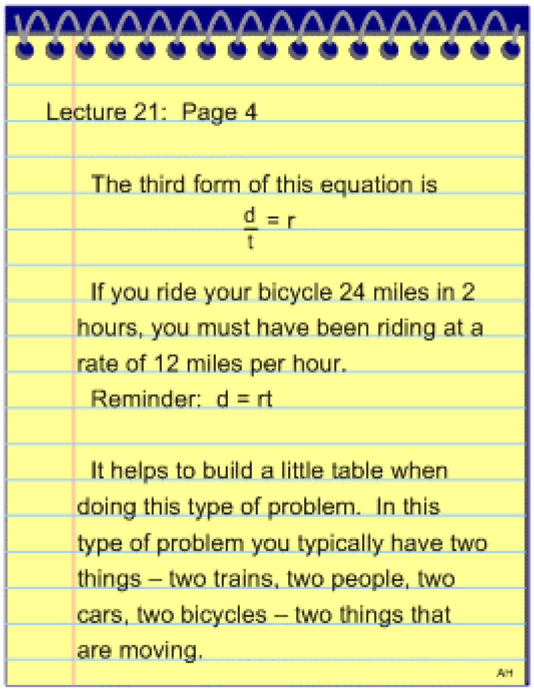
You would use the second equation to solve a problem like this:

If you're wanting to go 300 miles and you can only go 30 miles per hour, it's going to take you 10 hours.

Knowing any two of these things, you can always find the third one.

AH

CEMP021-04



Lecture 21: Page 4

The third form of this equation is

$$\frac{d}{t} = r$$

If you ride your bicycle 24 miles in 2 hours, you must have been riding at a rate of 12 miles per hour.

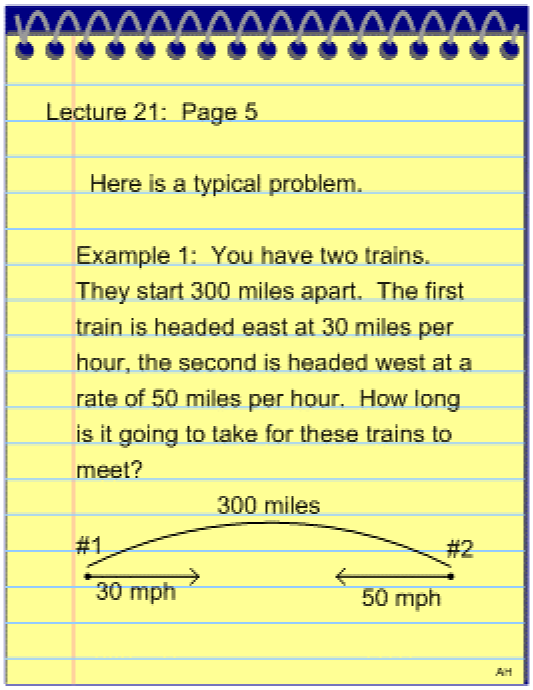
Reminder: $d = rt$

It helps to build a little table when doing this type of problem. In this type of problem you typically have two things – two trains, two people, two cars, two bicycles – two things that are moving.

AH

Lecture 21 Notes, Continued

CEMP021-05



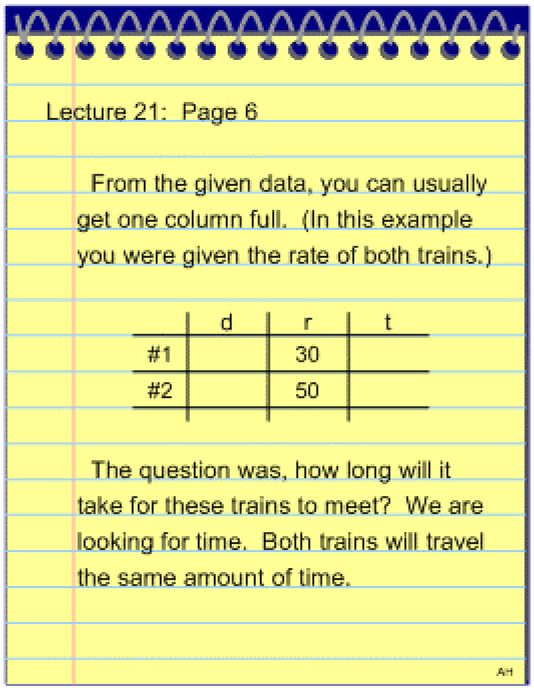
Lecture 21: Page 5

Here is a typical problem.

Example 1: You have two trains. They start 300 miles apart. The first train is headed east at 30 miles per hour, the second is headed west at a rate of 50 miles per hour. How long is it going to take for these trains to meet?

AH

CEMP021-06



Lecture 21: Page 6

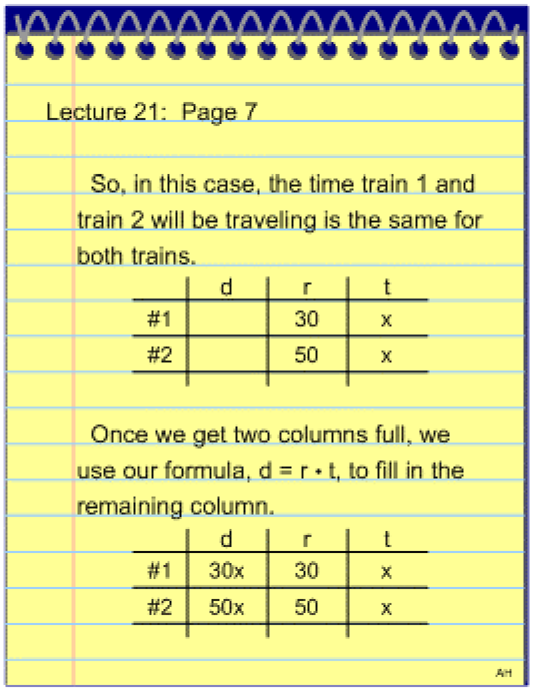
From the given data, you can usually get one column full. (In this example you were given the rate of both trains.)

	d	r	t
#1		30	
#2		50	

The question was, how long will it take for these trains to meet? We are looking for time. Both trains will travel the same amount of time.

AH

CEMP021-07



Lecture 21: Page 7

So, in this case, the time train 1 and train 2 will be traveling is the same for both trains.

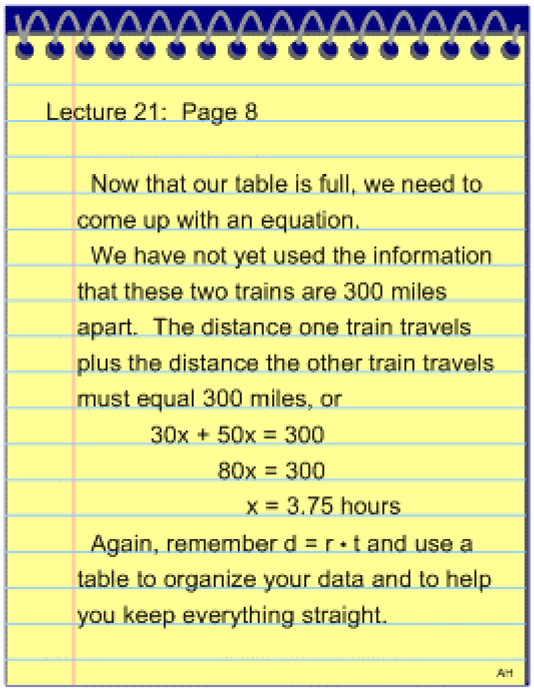
	d	r	t
#1		30	x
#2		50	x

Once we get two columns full, we use our formula, $d = r \cdot t$, to fill in the remaining column.

	d	r	t
#1	30x	30	x
#2	50x	50	x

AH

CEMP021-08



Lecture 21: Page 8

Now that our table is full, we need to come up with an equation.

We have not yet used the information that these two trains are 300 miles apart. The distance one train travels plus the distance the other train travels must equal 300 miles, or

$$30x + 50x = 300$$

$$80x = 300$$

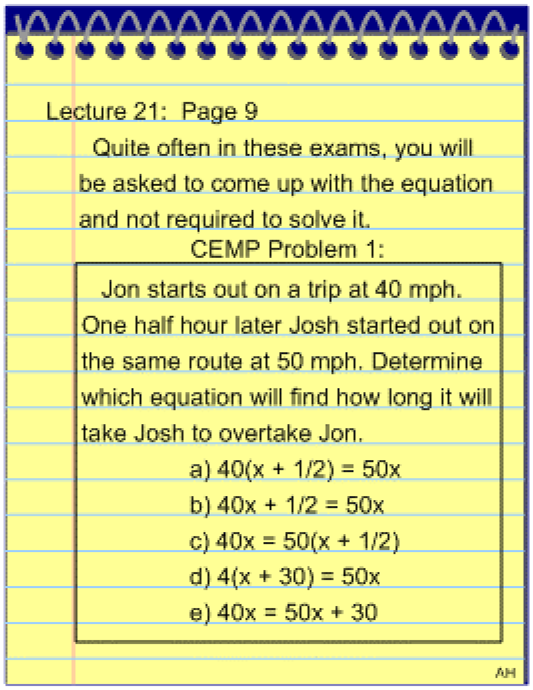
$$x = 3.75 \text{ hours}$$

Again, remember $d = r \cdot t$ and use a table to organize your data and to help you keep everything straight.

AH

Lecture 21 Notes, Continued

CEMP021-09



Lecture 21: Page 9

Quite often in these exams, you will be asked to come up with the equation and not required to solve it.

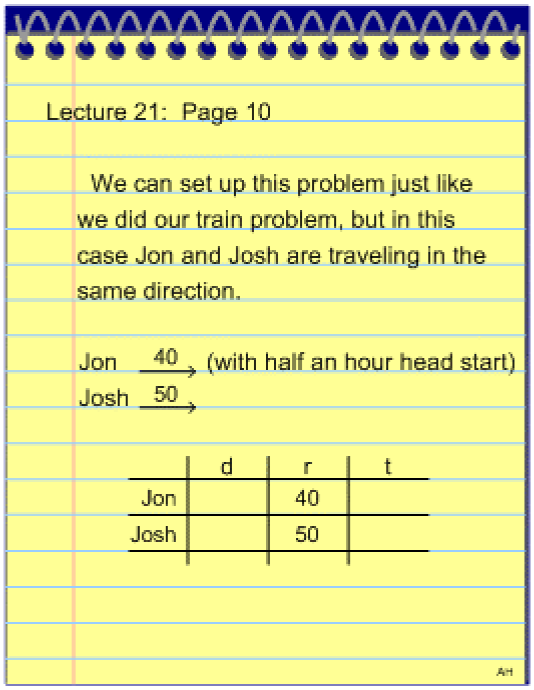
CEMP Problem 1:

Jon starts out on a trip at 40 mph. One half hour later Josh started out on the same route at 50 mph. Determine which equation will find how long it will take Josh to overtake Jon.

a) 40(x + 1/2) = 50x
b) 40x + 1/2 = 50x
c) 40x = 50(x + 1/2)
d) 4(x + 30) = 50x
e) 40x = 50x + 30

AH

CEMP021-10



Lecture 21: Page 10

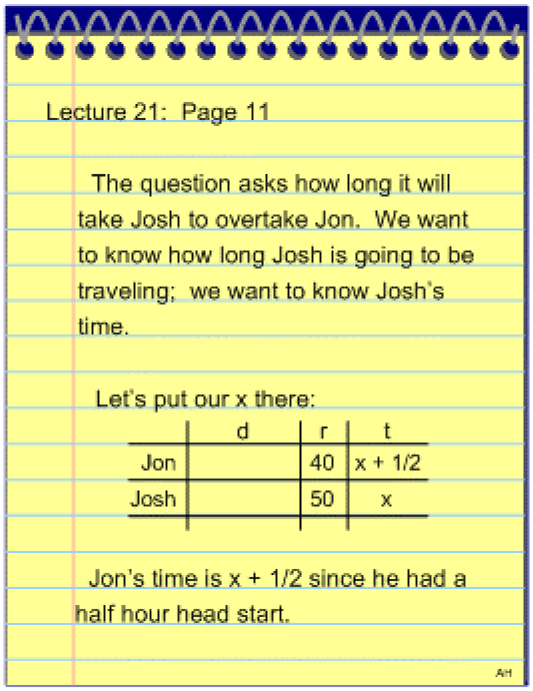
We can set up this problem just like we did our train problem, but in this case Jon and Josh are traveling in the same direction.

Jon $\xrightarrow{40}$ (with half an hour head start)
Josh $\xrightarrow{50}$

	d	r	t
Jon		40	
Josh		50	

AH

CEMP021-11



Lecture 21: Page 11

The question asks how long it will take Josh to overtake Jon. We want to know how long Josh is going to be traveling; we want to know Josh's time.

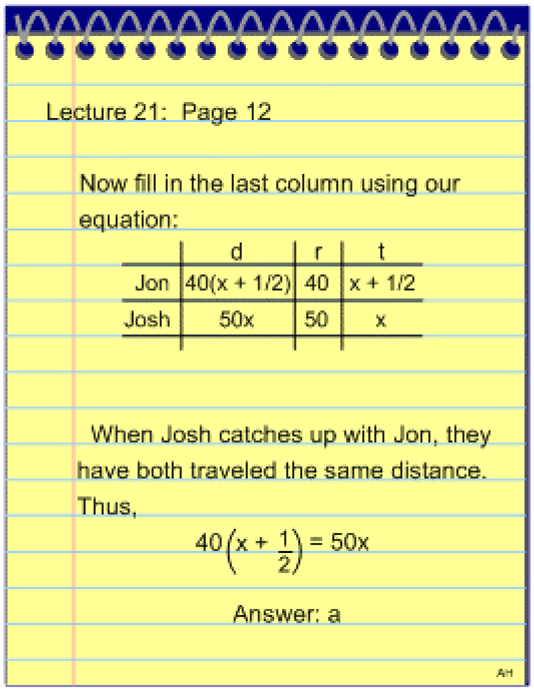
Let's put our x there:

	d	r	t
Jon		40	x + 1/2
Josh		50	x

Jon's time is x + 1/2 since he had a half hour head start.

AH

CEMP021-12



Lecture 21: Page 12

Now fill in the last column using our equation:

	d	r	t
Jon	40(x + 1/2)	40	x + 1/2
Josh	50x	50	x

When Josh catches up with Jon, they have both traveled the same distance. Thus,

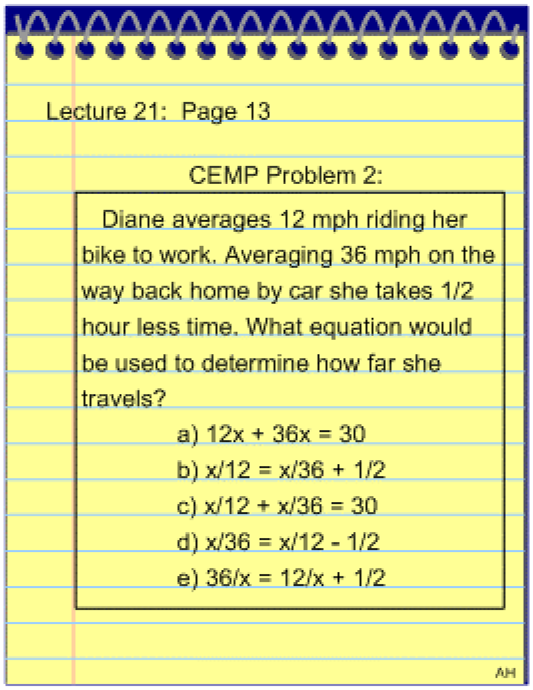
$$40\left(x + \frac{1}{2}\right) = 50x$$

Answer: a

AH

Lecture 21 Notes, Continued

CEMP021-13



Lecture 21: Page 13

CEMP Problem 2:

Diane averages 12 mph riding her bike to work. Averaging 36 mph on the way back home by car she takes 1/2 hour less time. What equation would be used to determine how far she travels?

a) 12x + 36x = 30

b) x/12 = x/36 + 1/2

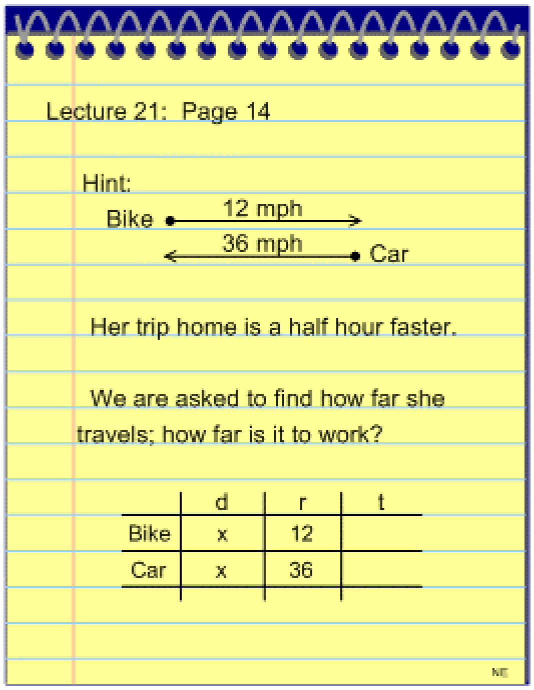
c) x/12 + x/36 = 30

d) x/36 = x/12 - 1/2

e) 36/x = 12/x + 1/2

AH

CEMP021-14



Lecture 21: Page 14

Hint:

Bike $\bullet\xrightarrow{12\text{ mph}}$

$\xleftarrow{36\text{ mph}}\bullet$ Car

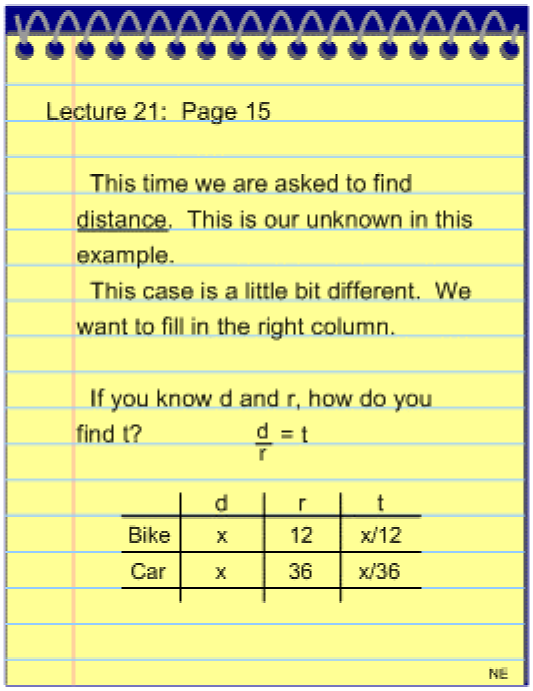
Her trip home is a half hour faster.

We are asked to find how far she travels; how far is it to work?

	d	r	t
Bike	x	12	
Car	x	36	

NE

CEMP021-15



Lecture 21: Page 15

This time we are asked to find distance. This is our unknown in this example.

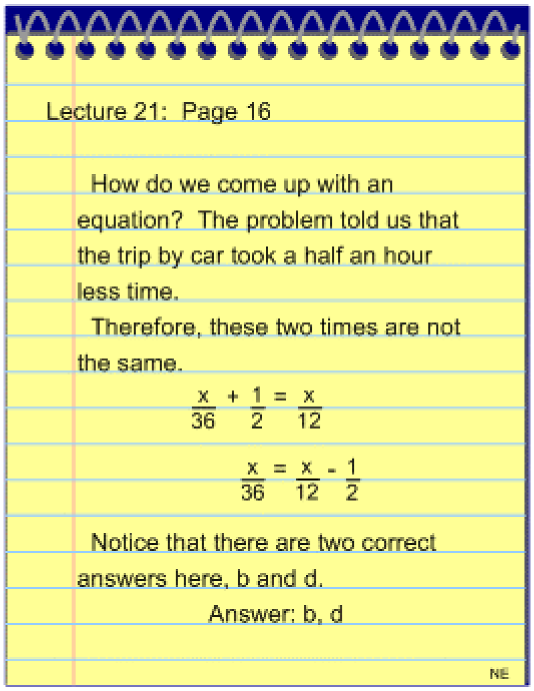
This case is a little bit different. We want to fill in the right column.

If you know d and r, how do you find t? $\frac{d}{r} = t$

	d	r	t
Bike	x	12	x/12
Car	x	36	x/36

NE

CEMP021-16



Lecture 21: Page 16

How do we come up with an equation? The problem told us that the trip by car took a half an hour less time.

Therefore, these two times are not the same.

$$\frac{x}{36} + \frac{1}{2} = \frac{x}{12}$$

$$\frac{x}{36} = \frac{x}{12} - \frac{1}{2}$$

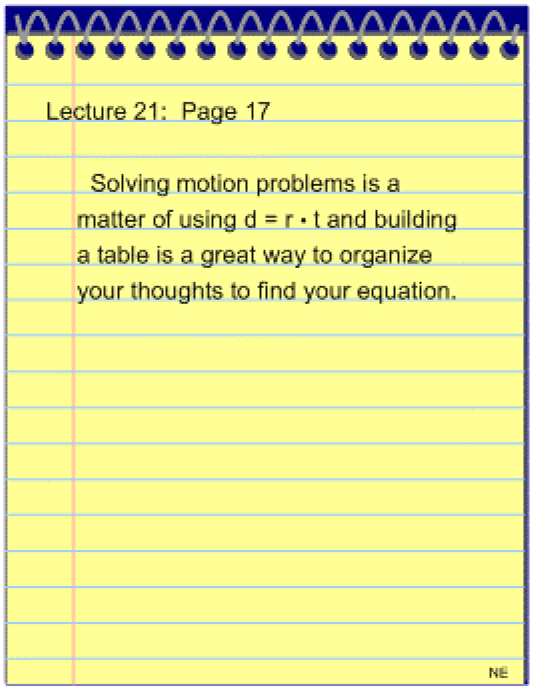
Notice that there are two correct answers here, b and d.

Answer: b, d

NE

Lecture 21 Notes, Continued

CEMP021-17

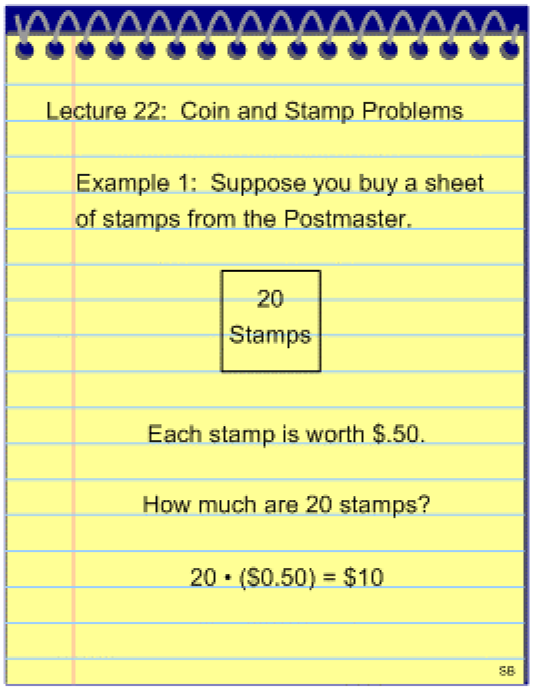


Lecture 21: Page 17

Solving motion problems is a matter of using d = r • t and building a table is a great way to organize your thoughts to find your equation.

Lecture 22 Notes

CEMP022-01



Lecture 22: Coin and Stamp Problems

Example 1: Suppose you buy a sheet of stamps from the Postmaster.

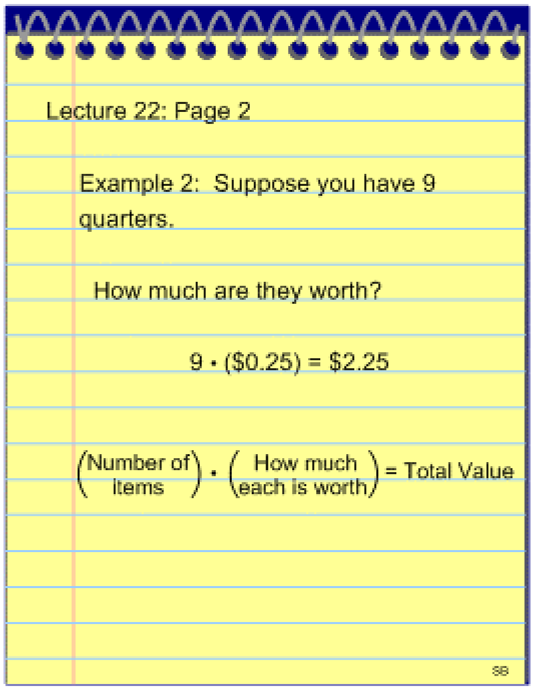
20 Stamps

Each stamp is worth $.50.

How much are 20 stamps?

20 • ($0.50) = $10

CEMP022-02



Lecture 22: Page 2

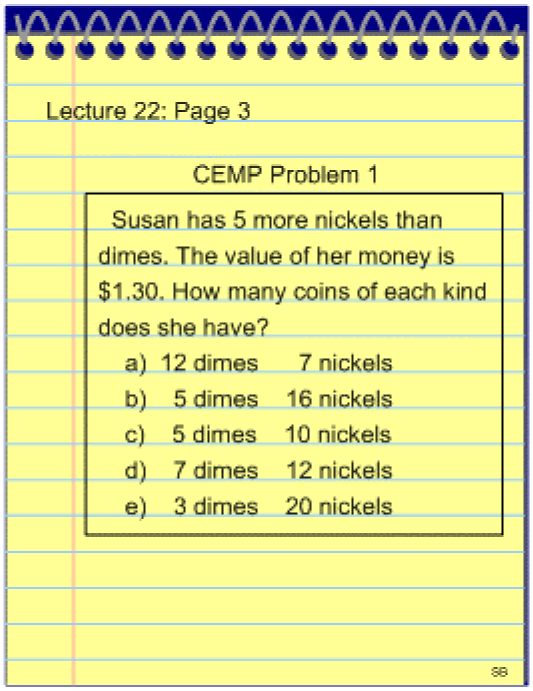
Example 2: Suppose you have 9 quarters.

How much are they worth?

9 • ($0.25) = $2.25

$\begin{pmatrix}\text{Number of}\\ \text{items}\end{pmatrix} \cdot \begin{pmatrix}\text{How much}\\ \text{each is worth}\end{pmatrix} = \text{Total Value}$

CEMP022-03



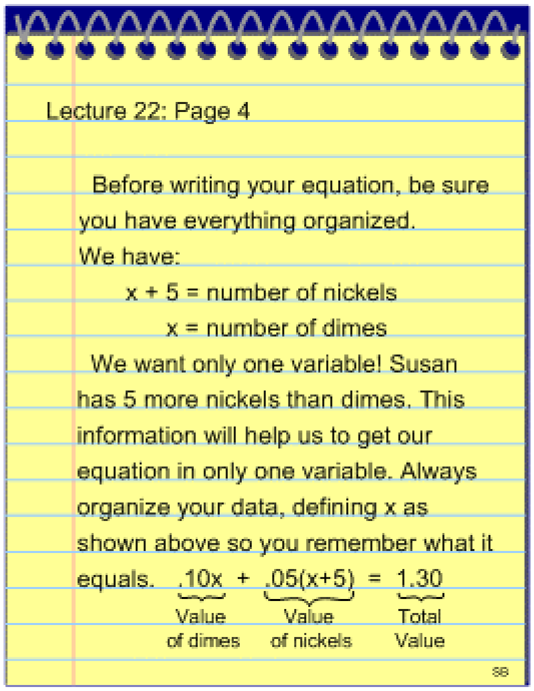
Lecture 22: Page 3

CEMP Problem 1

Susan has 5 more nickels than dimes. The value of her money is $1.30. How many coins of each kind does she have?

a) 12 dimes 7 nickels
b) 5 dimes 16 nickels
c) 5 dimes 10 nickels
d) 7 dimes 12 nickels
e) 3 dimes 20 nickels

CEMP022-04



Lecture 22: Page 4

Before writing your equation, be sure you have everything organized.

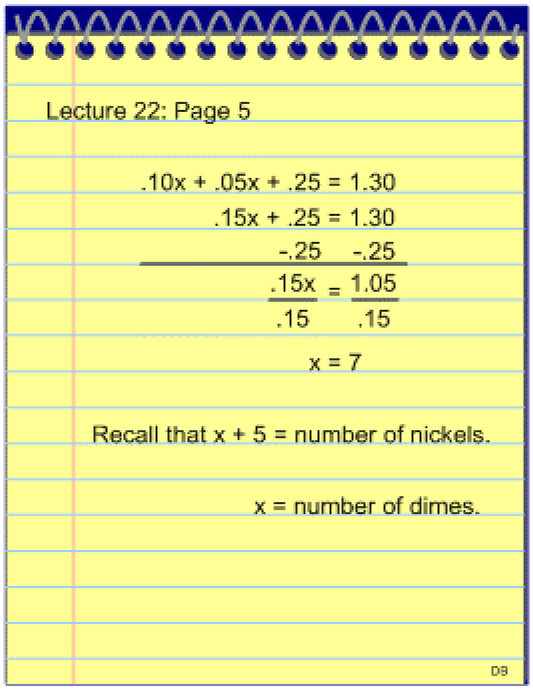
We have:

x + 5 = number of nickels
x = number of dimes

We want only one variable! Susan has 5 more nickels than dimes. This information will help us to get our equation in only one variable. Always organize your data, defining x as shown above so you remember what it equals. $\underbrace{.10x}_{\text{Value of dimes}} + \underbrace{.05(x+5)}_{\text{Value of nickels}} = \underbrace{1.30}_{\text{Total Value}}$

Lecture 22 Notes

CEMP022-05



Lecture 22: Page 5

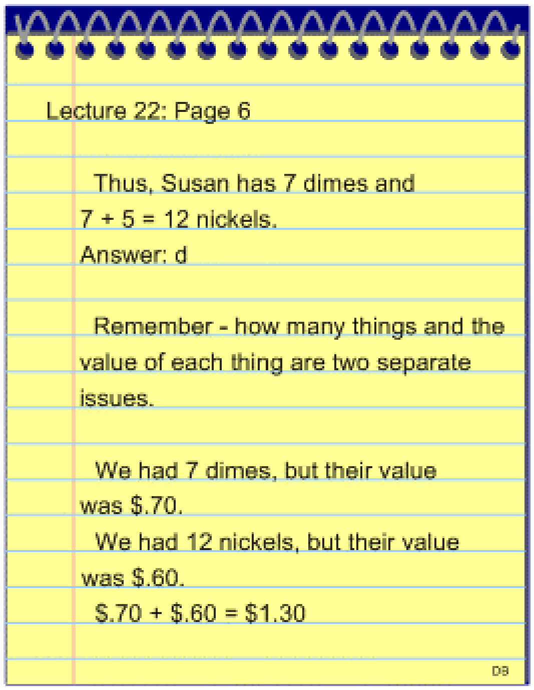
$$\begin{aligned} .10x + .05x + .25 &= 1.30 \\ .15x + .25 &= 1.30 \\ -.25 \quad &\;\; -.25 \\ \hline \frac{.15x}{.15} &= \frac{1.05}{.15} \\ x &= 7 \end{aligned}$$

Recall that x + 5 = number of nickels.

x = number of dimes.

D9

CEMP022-06



Lecture 22: Page 6

Thus, Susan has 7 dimes and
7 + 5 = 12 nickels.
Answer: d

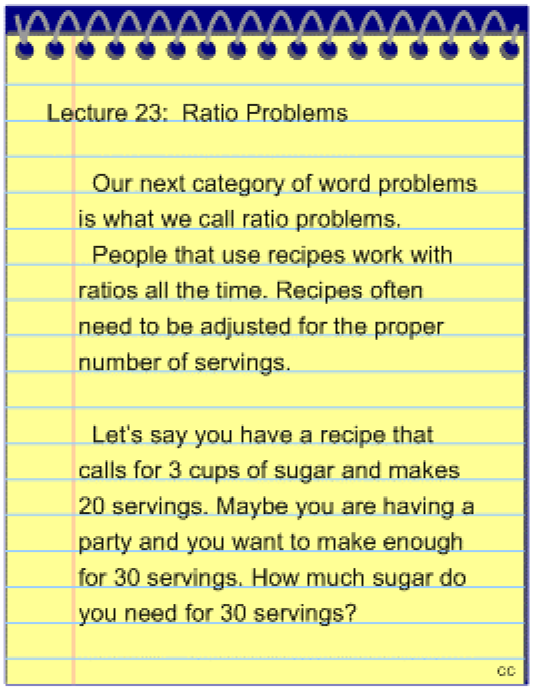
Remember - how many things and the value of each thing are two separate issues.

We had 7 dimes, but their value was \$.70.
We had 12 nickels, but their value was \$.60.
\$.70 + \$.60 = \$1.30

D9

Lecture 23 Notes

CEMP023-01



Lecture 23: Ratio Problems

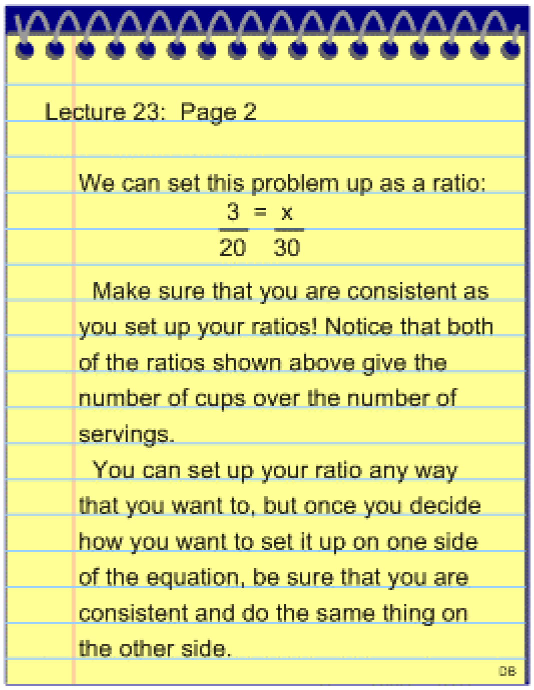
Our next category of word problems is what we call ratio problems.

People that use recipes work with ratios all the time. Recipes often need to be adjusted for the proper number of servings.

Let's say you have a recipe that calls for 3 cups of sugar and makes 20 servings. Maybe you are having a party and you want to make enough for 30 servings. How much sugar do you need for 30 servings?

cc

CEMP023-02



Lecture 23: Page 2

We can set this problem up as a ratio:

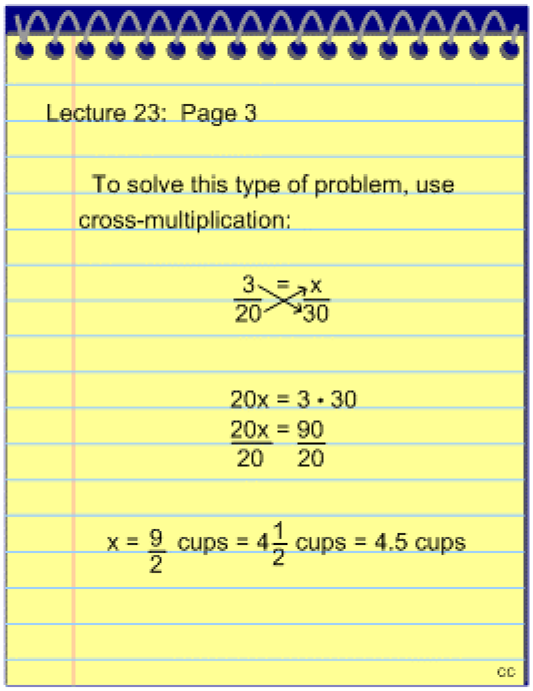
$$\frac{3}{20} = \frac{x}{30}$$

Make sure that you are consistent as you set up your ratios! Notice that both of the ratios shown above give the number of cups over the number of servings.

You can set up your ratio any way that you want to, but once you decide how you want to set it up on one side of the equation, be sure that you are consistent and do the same thing on the other side.

DB

CEMP023-03



Lecture 23: Page 3

To solve this type of problem, use cross-multiplication:

$$\frac{3}{20} \times \frac{x}{30}$$

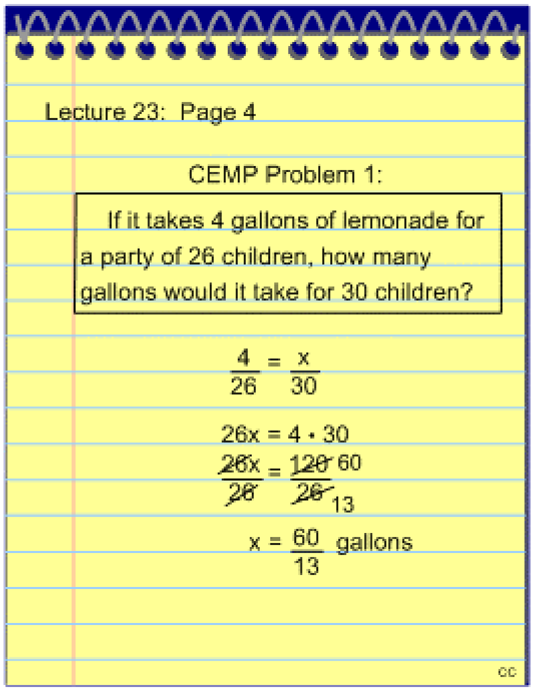
$$20x = 3 \cdot 30$$

$$\frac{20x}{20} = \frac{90}{20}$$

$$x = \frac{9}{2} \text{ cups} = 4\tfrac{1}{2} \text{ cups} = 4.5 \text{ cups}$$

cc

CEMP023-04



Lecture 23: Page 4

CEMP Problem 1:

If it takes 4 gallons of lemonade for a party of 26 children, how many gallons would it take for 30 children?

$$\frac{4}{26} = \frac{x}{30}$$

$$26x = 4 \cdot 30$$

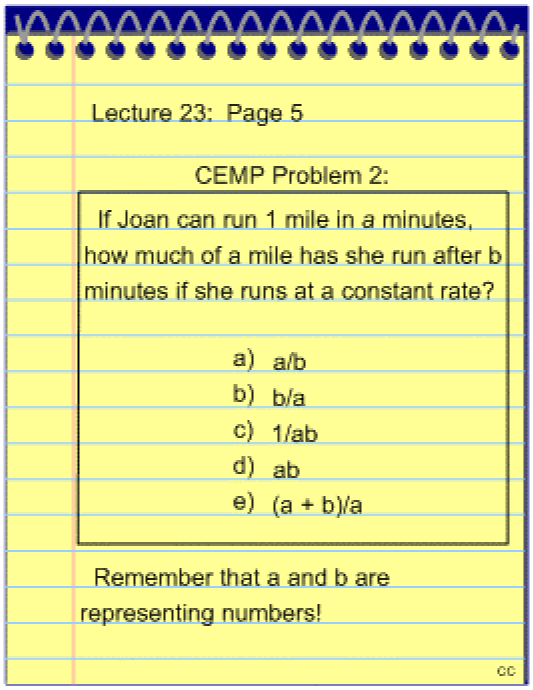
$$\frac{\cancel{26}x}{\cancel{26}} = \frac{\cancel{120}^{60}}{\cancel{26}_{13}}$$

$$x = \frac{60}{13} \text{ gallons}$$

cc

Lecture 23 Notes, Continued

CEMP023-05



Lecture 23: Page 5

CEMP Problem 2:

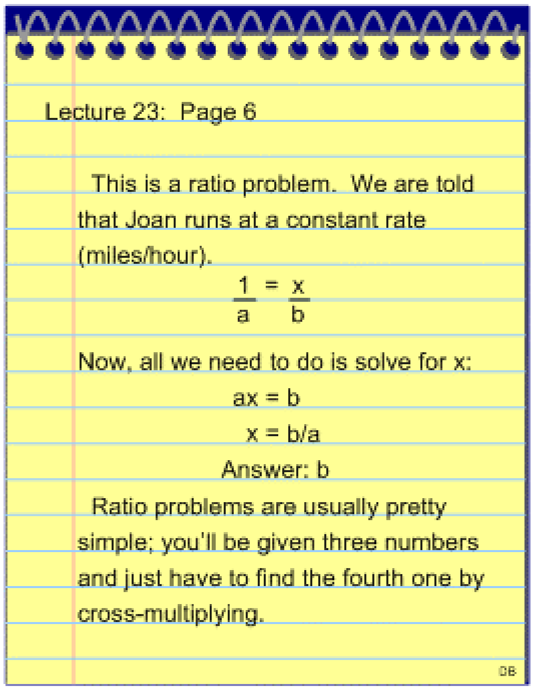
If Joan can run 1 mile in *a* minutes, how much of a mile has she run after b minutes if she runs at a constant rate?

a) a/b
b) b/a
c) 1/ab
d) ab
e) (a + b)/a

Remember that a and b are representing numbers!

CC

CEMP023-06



Lecture 23: Page 6

This is a ratio problem. We are told that Joan runs at a constant rate (miles/hour).

$$\frac{1}{a} = \frac{x}{b}$$

Now, all we need to do is solve for x:

$$ax = b$$

$$x = b/a$$

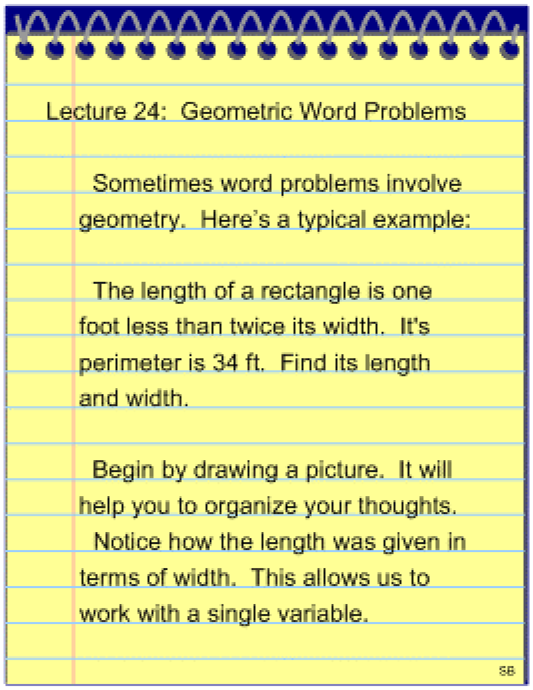
Answer: b

Ratio problems are usually pretty simple; you'll be given three numbers and just have to find the fourth one by cross-multiplying.

DB

Lecture 24 Notes

CEMP024-01



Lecture 24: Geometric Word Problems

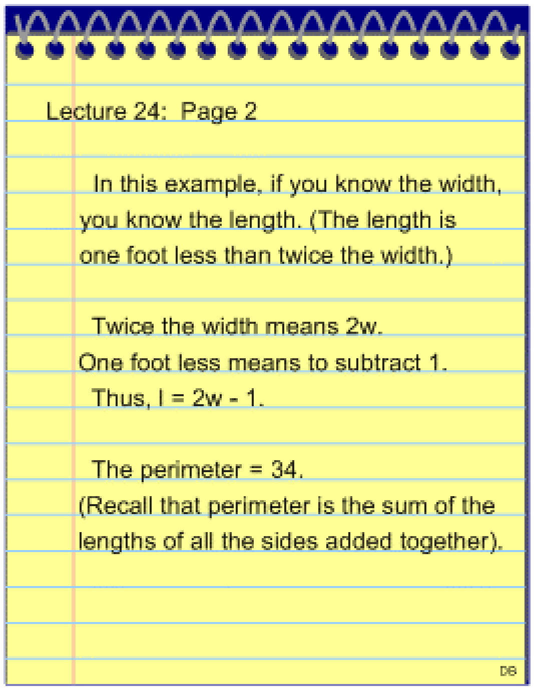
Sometimes word problems involve geometry. Here's a typical example:

The length of a rectangle is one foot less than twice its width. It's perimeter is 34 ft. Find its length and width.

Begin by drawing a picture. It will help you to organize your thoughts. Notice how the length was given in terms of width. This allows us to work with a single variable.

SB

CEMP024-02



Lecture 24: Page 2

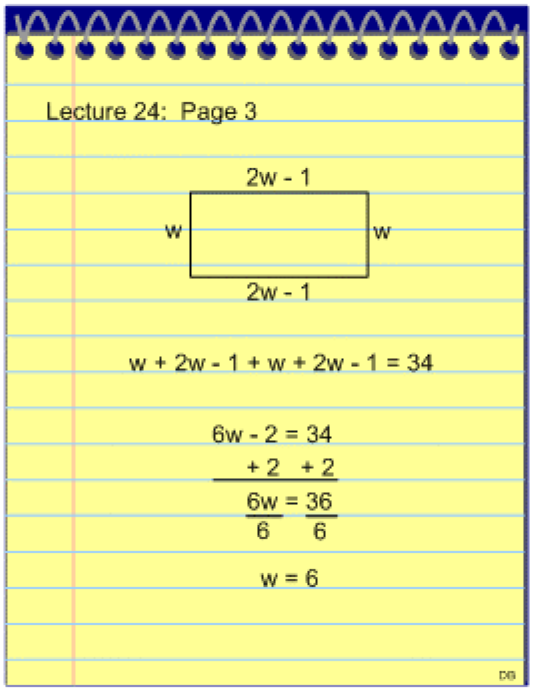
In this example, if you know the width, you know the length. (The length is one foot less than twice the width.)

Twice the width means 2w.
One foot less means to subtract 1.
Thus, l = 2w - 1.

The perimeter = 34.
(Recall that perimeter is the sum of the lengths of all the sides added together).

DB

CEMP024-03



Lecture 24: Page 3

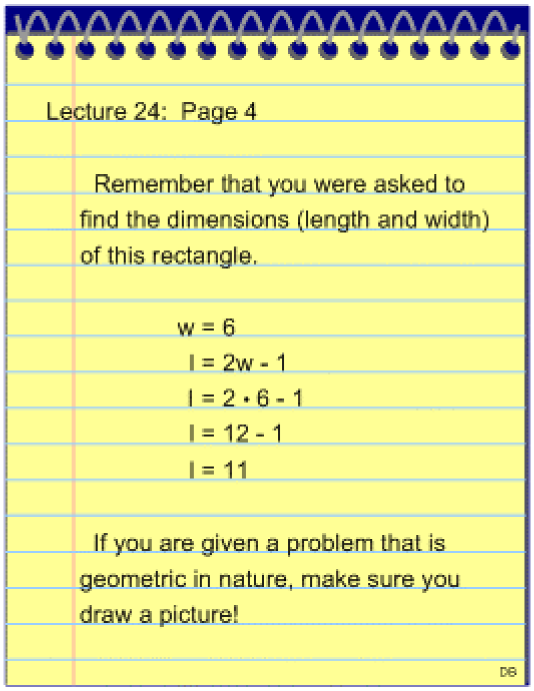
$$w + 2w - 1 + w + 2w - 1 = 34$$

$$\begin{array}{r} 6w - 2 = 34 \\ \underline{+2 \quad +2} \\ \frac{6w}{6} = \frac{36}{6} \end{array}$$

$$w = 6$$

DB

CEMP024-04



Lecture 24: Page 4

Remember that you were asked to find the dimensions (length and width) of this rectangle.

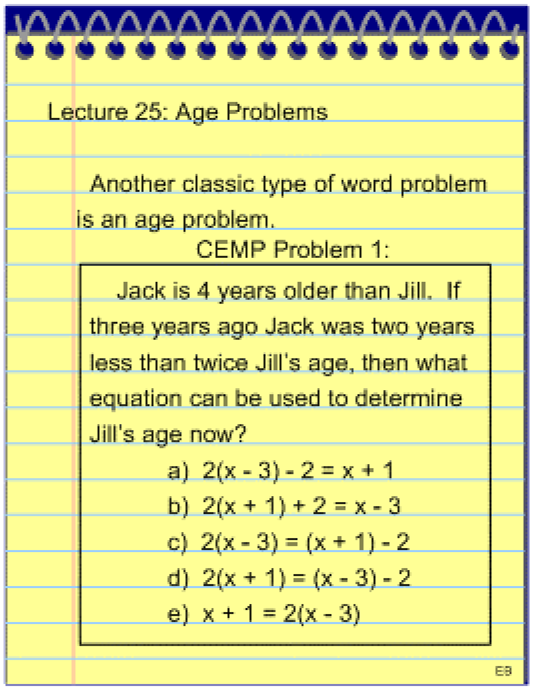
$w = 6$
$l = 2w - 1$
$l = 2 \cdot 6 - 1$
$l = 12 - 1$
$l = 11$

If you are given a problem that is geometric in nature, make sure you draw a picture!

DB

Lecture 25 Notes

CEMP025-01



Lecture 25: Age Problems

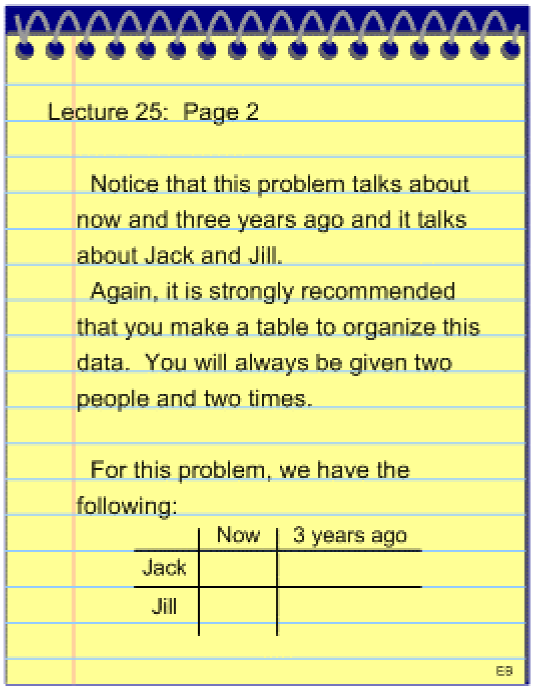
Another classic type of word problem is an age problem.

CEMP Problem 1:

Jack is 4 years older than Jill. If three years ago Jack was two years less than twice Jill's age, then what equation can be used to determine Jill's age now?

a) $2(x - 3) - 2 = x + 1$
b) $2(x + 1) + 2 = x - 3$
c) $2(x - 3) = (x + 1) - 2$
d) $2(x + 1) = (x - 3) - 2$
e) $x + 1 = 2(x - 3)$

CEMP025-02



Lecture 25: Page 2

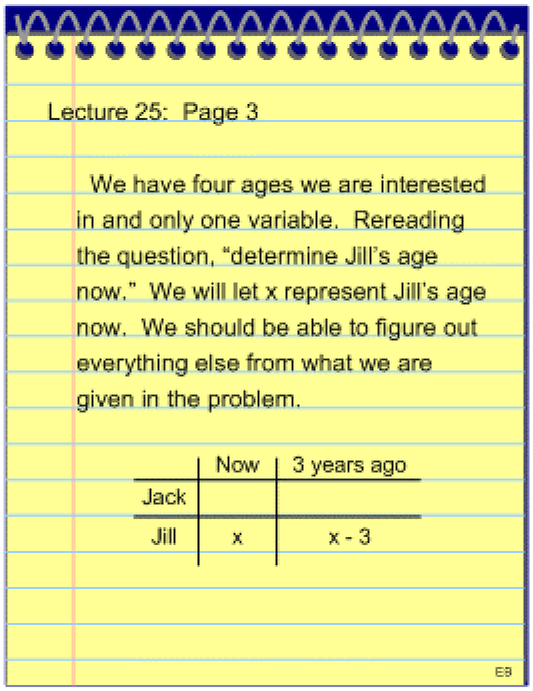
Notice that this problem talks about now and three years ago and it talks about Jack and Jill.

Again, it is strongly recommended that you make a table to organize this data. You will always be given two people and two times.

For this problem, we have the following:

	Now	3 years ago
Jack		
Jill		

CEMP025-03

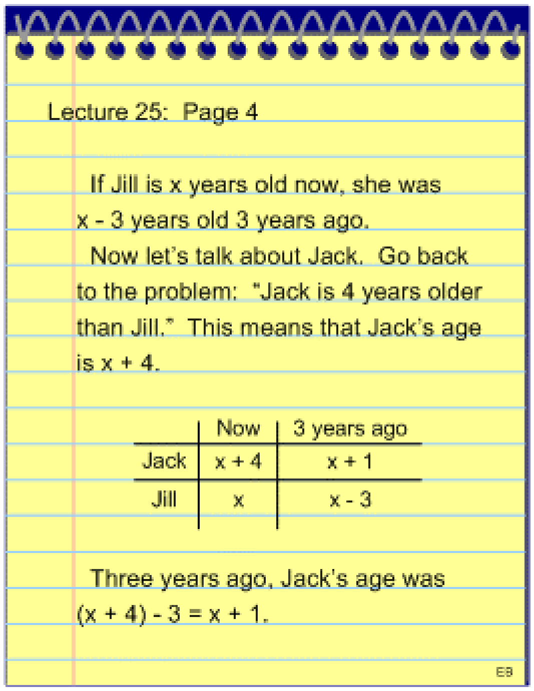


Lecture 25: Page 3

We have four ages we are interested in and only one variable. Rereading the question, "determine Jill's age now." We will let x represent Jill's age now. We should be able to figure out everything else from what we are given in the problem.

	Now	3 years ago
Jack		
Jill	x	$x - 3$

CEMP025-04



Lecture 25: Page 4

If Jill is x years old now, she was $x - 3$ years old 3 years ago.

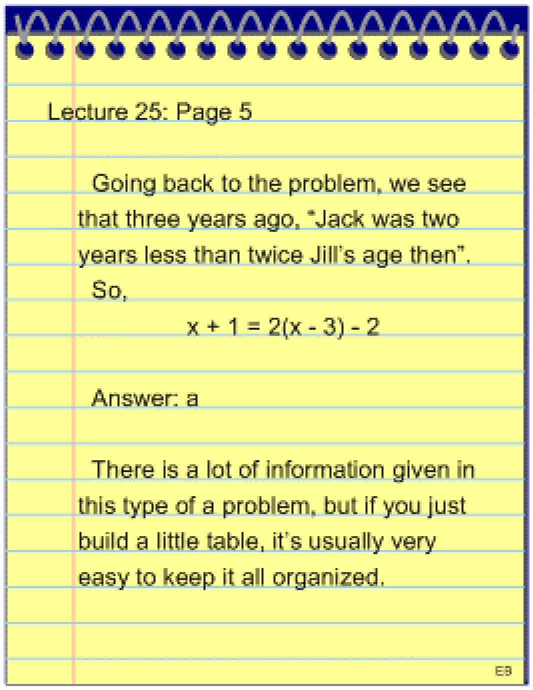
Now let's talk about Jack. Go back to the problem: "Jack is 4 years older than Jill." This means that Jack's age is $x + 4$.

	Now	3 years ago
Jack	$x + 4$	$x + 1$
Jill	x	$x - 3$

Three years ago, Jack's age was $(x + 4) - 3 = x + 1$.

Lecture 25 Notes, Continued

CEMP025-05



Lecture 25: Page 5

Going back to the problem, we see that three years ago, "Jack was two years less than twice Jill's age then".

So,

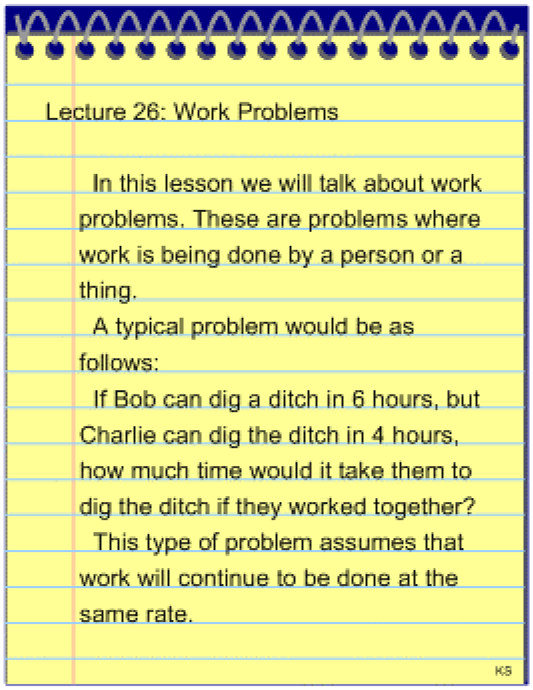
$$x + 1 = 2(x - 3) - 2$$

Answer: a

There is a lot of information given in this type of a problem, but if you just build a little table, it's usually very easy to keep it all organized.

Lecture 26 Notes

CEMP026-01



Lecture 26: Work Problems

In this lesson we will talk about work problems. These are problems where work is being done by a person or a thing.

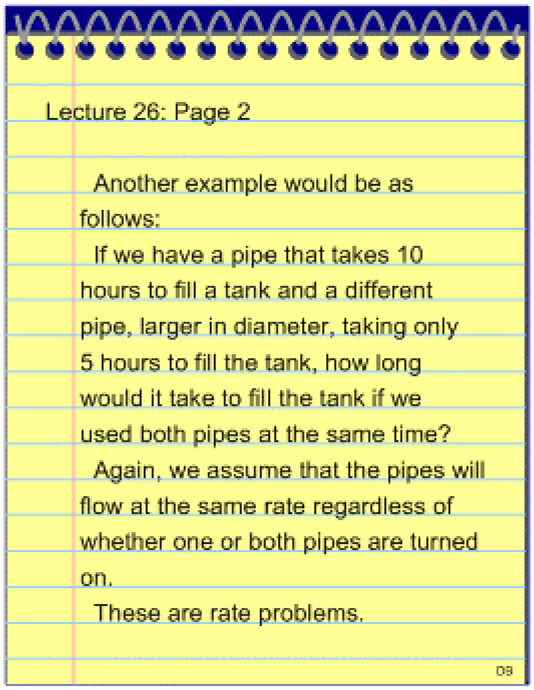
A typical problem would be as follows:

If Bob can dig a ditch in 6 hours, but Charlie can dig the ditch in 4 hours, how much time would it take them to dig the ditch if they worked together?

This type of problem assumes that work will continue to be done at the same rate.

KS

CEMP026-02



Lecture 26: Page 2

Another example would be as follows:

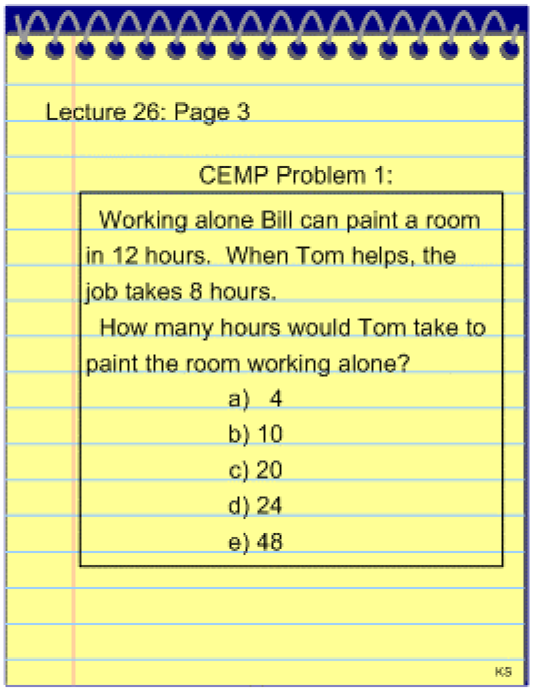
If we have a pipe that takes 10 hours to fill a tank and a different pipe, larger in diameter, taking only 5 hours to fill the tank, how long would it take to fill the tank if we used both pipes at the same time?

Again, we assume that the pipes will flow at the same rate regardless of whether one or both pipes are turned on.

These are rate problems.

DB

CEMP026-03



Lecture 26: Page 3

CEMP Problem 1:

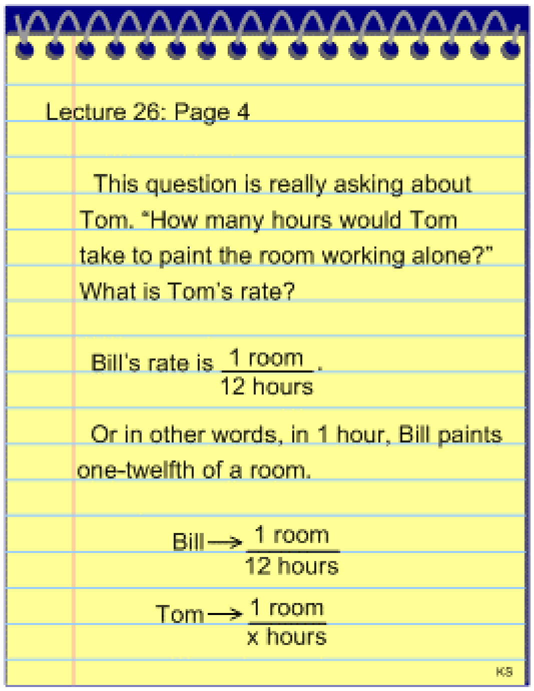
Working alone Bill can paint a room in 12 hours. When Tom helps, the job takes 8 hours.

How many hours would Tom take to paint the room working alone?

a) 4
b) 10
c) 20
d) 24
e) 48

KS

CEMP026-04



Lecture 26: Page 4

This question is really asking about Tom. "How many hours would Tom take to paint the room working alone?" What is Tom's rate?

Bill's rate is $\frac{1 \text{ room}}{12 \text{ hours}}$.

Or in other words, in 1 hour, Bill paints one-twelfth of a room.

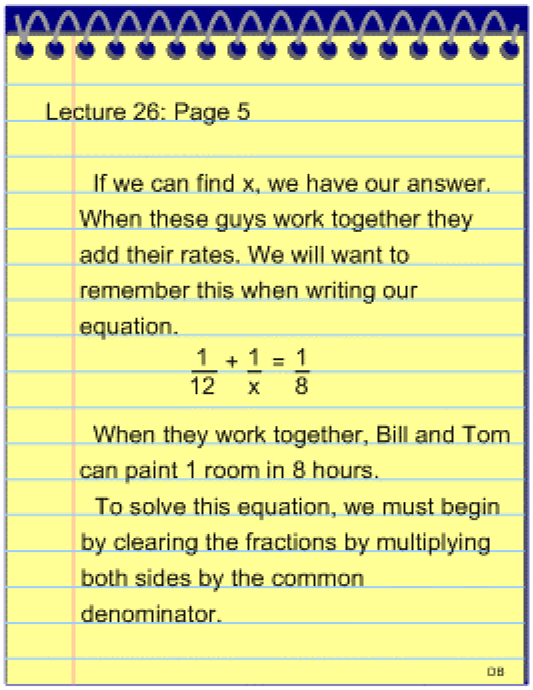
$$\text{Bill} \rightarrow \frac{1 \text{ room}}{12 \text{ hours}}$$

$$\text{Tom} \rightarrow \frac{1 \text{ room}}{x \text{ hours}}$$

KS

Lecture 26 Notes

CEMP026-05



Lecture 26: Page 5

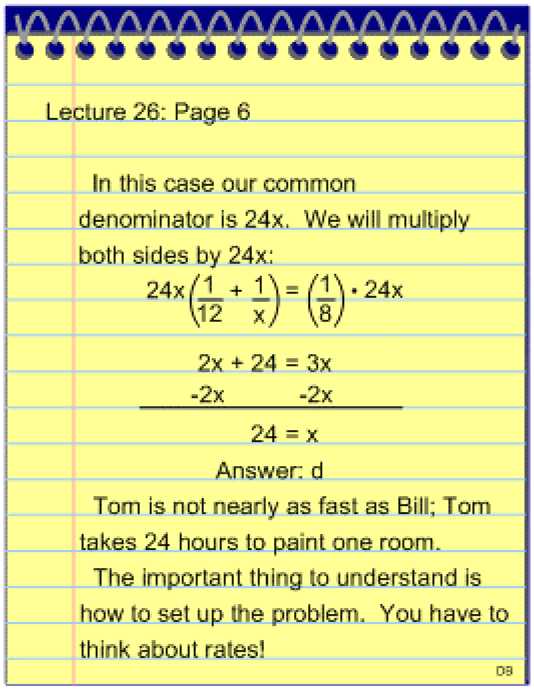
If we can find x, we have our answer. When these guys work together they add their rates. We will want to remember this when writing our equation.

$$\frac{1}{12}+\frac{1}{x}=\frac{1}{8}$$

When they work together, Bill and Tom can paint 1 room in 8 hours.

To solve this equation, we must begin by clearing the fractions by multiplying both sides by the common denominator.

CEMP026-06



Lecture 26: Page 6

In this case our common denominator is 24x. We will multiply both sides by 24x:

$$24x\left(\frac{1}{12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)\cdot 24x$$

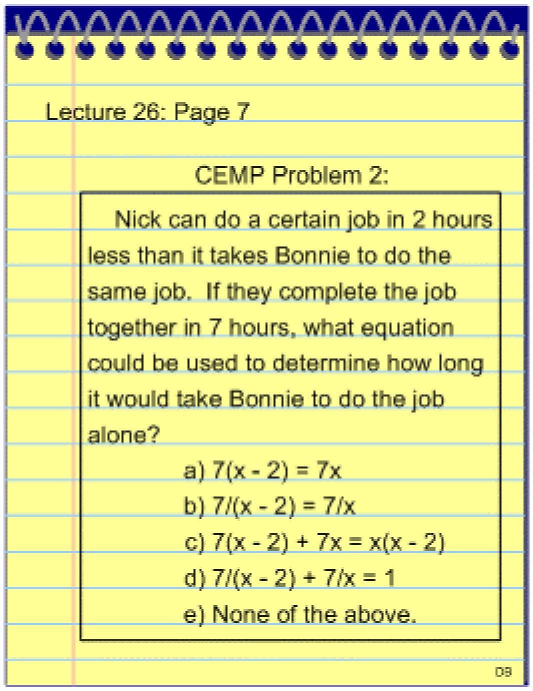
$$\begin{array}{rcl} 2x + 24 & = & 3x \\ \underline{-2x} & & \underline{-2x} \\ 24 & = & x \end{array}$$

Answer: d

Tom is not nearly as fast as Bill; Tom takes 24 hours to paint one room.

The important thing to understand is how to set up the problem. You have to think about rates!

CEMP026-07



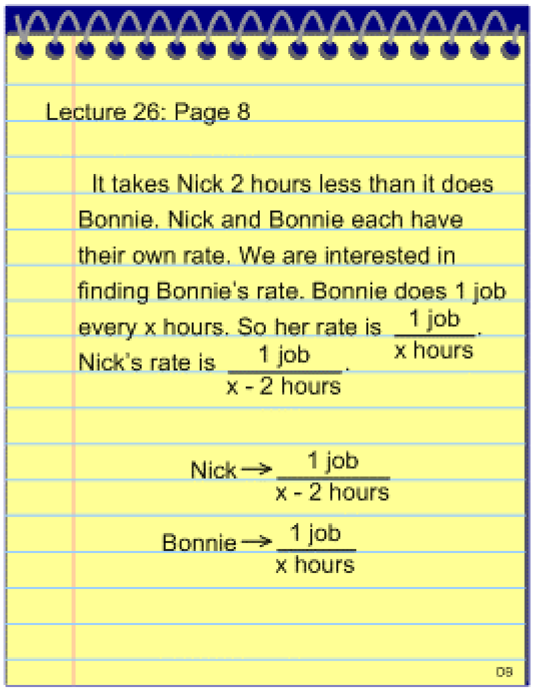
Lecture 26: Page 7

CEMP Problem 2:

Nick can do a certain job in 2 hours less than it takes Bonnie to do the same job. If they complete the job together in 7 hours, what equation could be used to determine how long it would take Bonnie to do the job alone?

a) 7(x - 2) = 7x
b) 7/(x - 2) = 7/x
c) 7(x - 2) + 7x = x(x - 2)
d) 7/(x - 2) + 7/x = 1
e) None of the above.

CEMP026-08



Lecture 26: Page 8

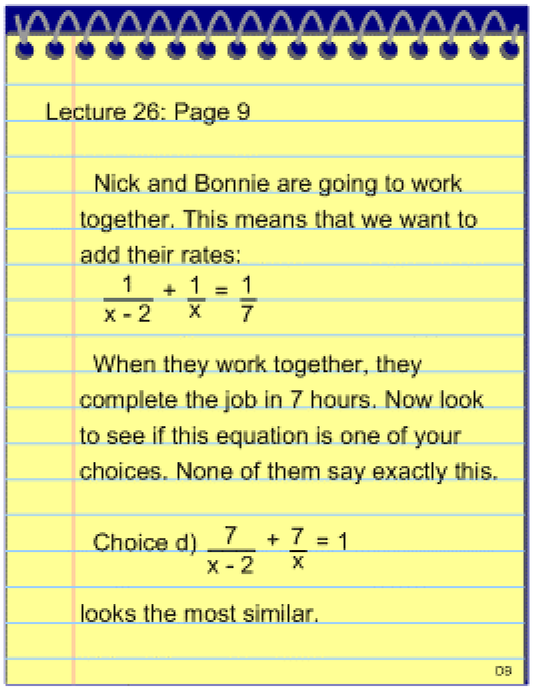
It takes Nick 2 hours less than it does Bonnie. Nick and Bonnie each have their own rate. We are interested in finding Bonnie's rate. Bonnie does 1 job every x hours. So her rate is $\frac{1 \text{ job}}{x \text{ hours}}$. Nick's rate is $\frac{1 \text{ job}}{x - 2 \text{ hours}}$.

$$\text{Nick} \rightarrow \frac{1 \text{ job}}{x - 2 \text{ hours}}$$

$$\text{Bonnie} \rightarrow \frac{1 \text{ job}}{x \text{ hours}}$$

Lecture 26 Notes

CEMP026-09



Lecture 26: Page 9

Nick and Bonnie are going to work together. This means that we want to add their rates:

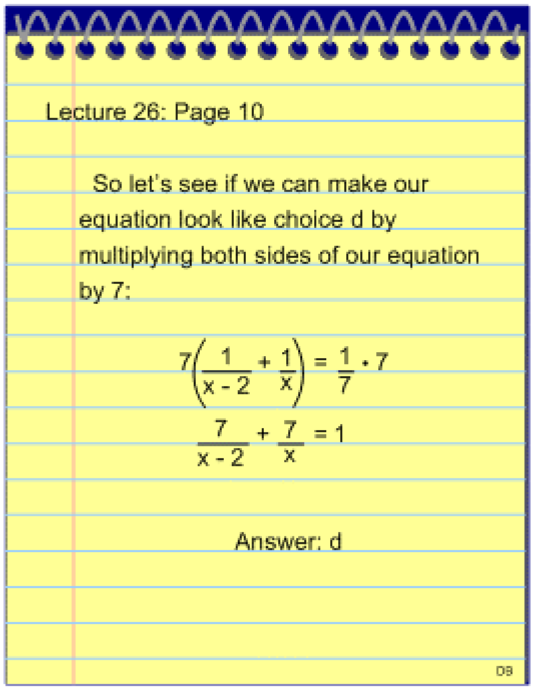
$$\frac{1}{x-2} + \frac{1}{x} = \frac{1}{7}$$

When they work together, they complete the job in 7 hours. Now look to see if this equation is one of your choices. None of them say exactly this.

Choice d) $\frac{7}{x-2} + \frac{7}{x} = 1$

looks the most similar.

CEMP026-10



Lecture 26: Page 10

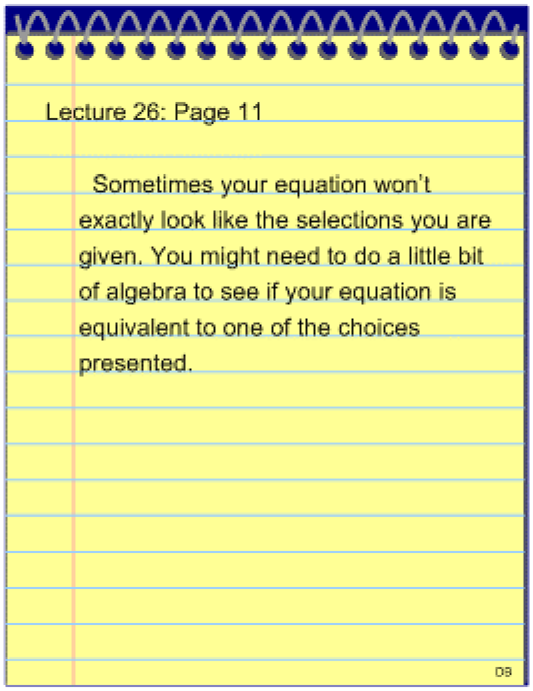
So let's see if we can make our equation look like choice d by multiplying both sides of our equation by 7:

$$7\left(\frac{1}{x-2} + \frac{1}{x}\right) = \frac{1}{7} \cdot 7$$

$$\frac{7}{x-2} + \frac{7}{x} = 1$$

Answer: d

CEMP026-11

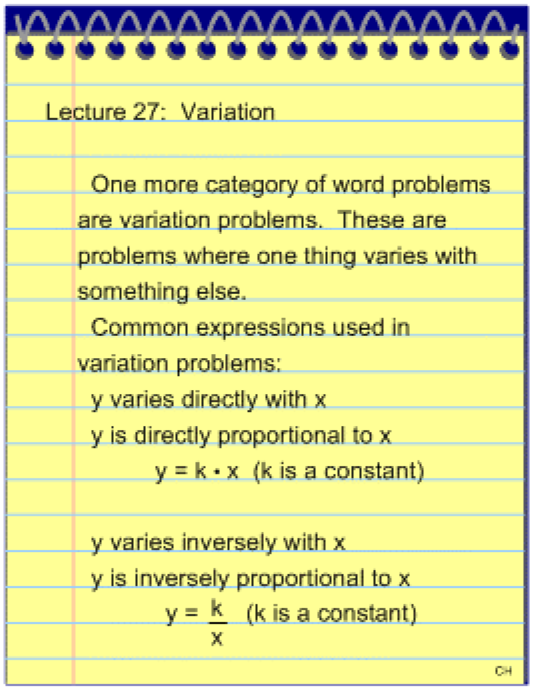


Lecture 26: Page 11

Sometimes your equation won't exactly look like the selections you are given. You might need to do a little bit of algebra to see if your equation is equivalent to one of the choices presented.

Lecture 27 Notes

CEMP027-01



Lecture 27: Variation

One more category of word problems are variation problems. These are problems where one thing varies with something else.

Common expressions used in variation problems:

y varies directly with x

y is directly proportional to x

$y = k \cdot x$ (k is a constant)

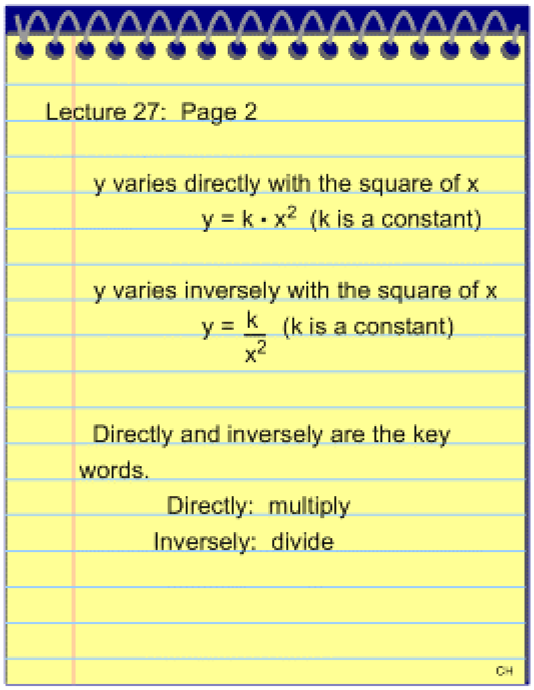
y varies inversely with x

y is inversely proportional to x

$y = \frac{k}{x}$ (k is a constant)

CH

CEMP027-02



Lecture 27: Page 2

y varies directly with the square of x

$y = k \cdot x^2$ (k is a constant)

y varies inversely with the square of x

$y = \frac{k}{x^2}$ (k is a constant)

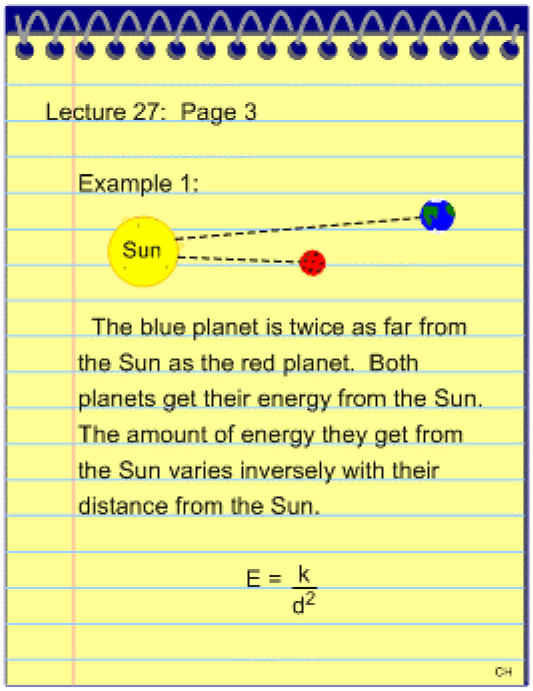
Directly and inversely are the key words.

Directly: multiply

Inversely: divide

CH

CEMP027-03



Lecture 27: Page 3

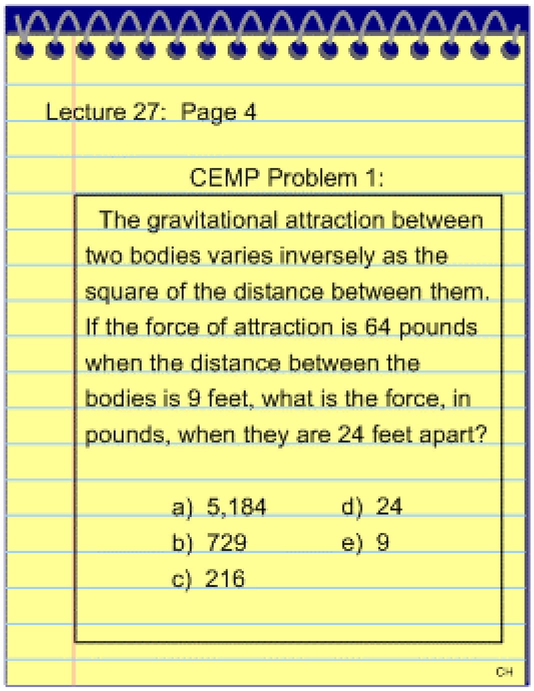
Example 1:

The blue planet is twice as far from the Sun as the red planet. Both planets get their energy from the Sun. The amount of energy they get from the Sun varies inversely with their distance from the Sun.

$E = \frac{k}{d^2}$

CH

CEMP027-04



Lecture 27: Page 4

CEMP Problem 1:

The gravitational attraction between two bodies varies inversely as the square of the distance between them. If the force of attraction is 64 pounds when the distance between the bodies is 9 feet, what is the force, in pounds, when they are 24 feet apart?

a) 5,184

b) 729

c) 216

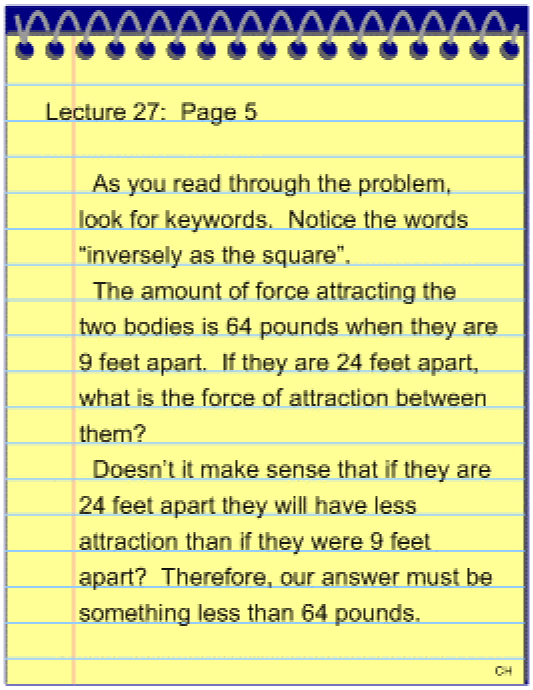
d) 24

e) 9

CH

Lecture 27 Notes, Continued

CEMP027-05



Lecture 27: Page 5

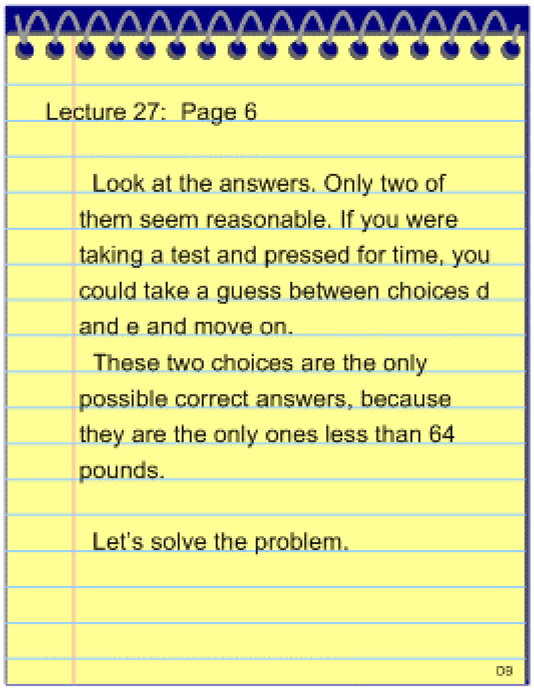
As you read through the problem, look for keywords. Notice the words "inversely as the square".

The amount of force attracting the two bodies is 64 pounds when they are 9 feet apart. If they are 24 feet apart, what is the force of attraction between them?

Doesn't it make sense that if they are 24 feet apart they will have less attraction than if they were 9 feet apart? Therefore, our answer must be something less than 64 pounds.

CH

CEMP027-06



Lecture 27: Page 6

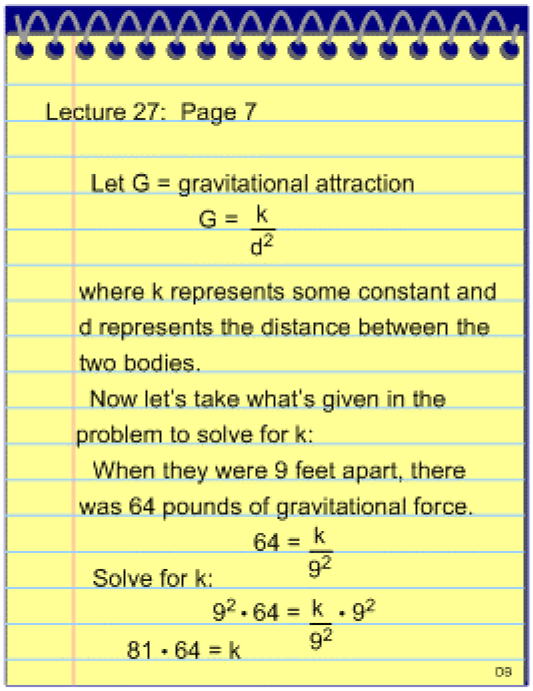
Look at the answers. Only two of them seem reasonable. If you were taking a test and pressed for time, you could take a guess between choices d and e and move on.

These two choices are the only possible correct answers, because they are the only ones less than 64 pounds.

Let's solve the problem.

D3

CEMP027-07



Lecture 27: Page 7

Let G = gravitational attraction

$$G = \frac{k}{d^2}$$

where k represents some constant and d represents the distance between the two bodies.

Now let's take what's given in the problem to solve for k:

When they were 9 feet apart, there was 64 pounds of gravitational force.

$$64 = \frac{k}{9^2}$$

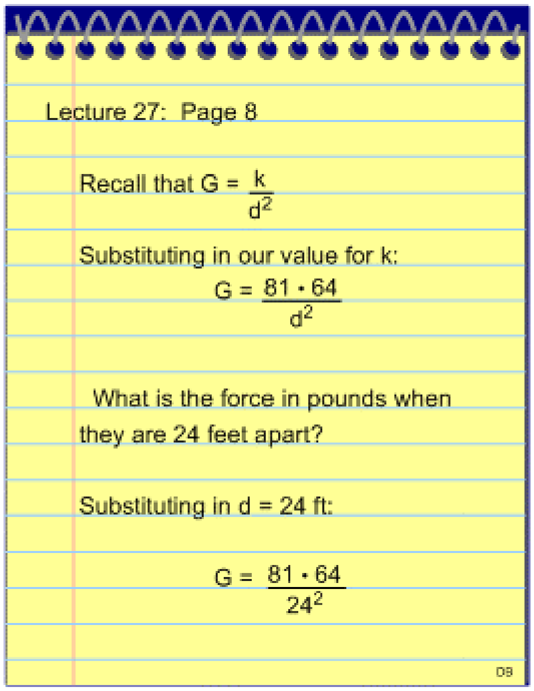
Solve for k:

$$9^2 \cdot 64 = \frac{k}{9^2} \cdot 9^2$$

$$81 \cdot 64 = k$$

D3

CEMP027-08



Lecture 27: Page 8

Recall that $G = \frac{k}{d^2}$

Substituting in our value for k:

$$G = \frac{81 \cdot 64}{d^2}$$

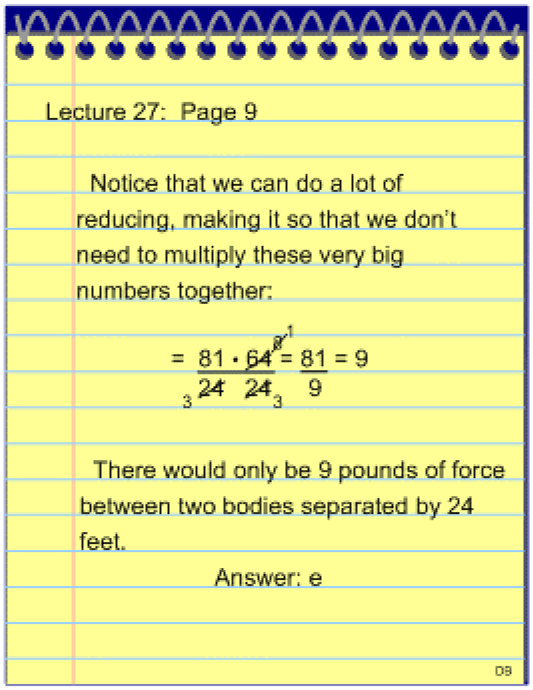
What is the force in pounds when they are 24 feet apart?

Substituting in d = 24 ft:

$$G = \frac{81 \cdot 64}{24^2}$$

D3

CEMP027-09



Lecture 27: Page 9

Notice that we can do a lot of reducing, making it so that we don't need to multiply these very big numbers together:

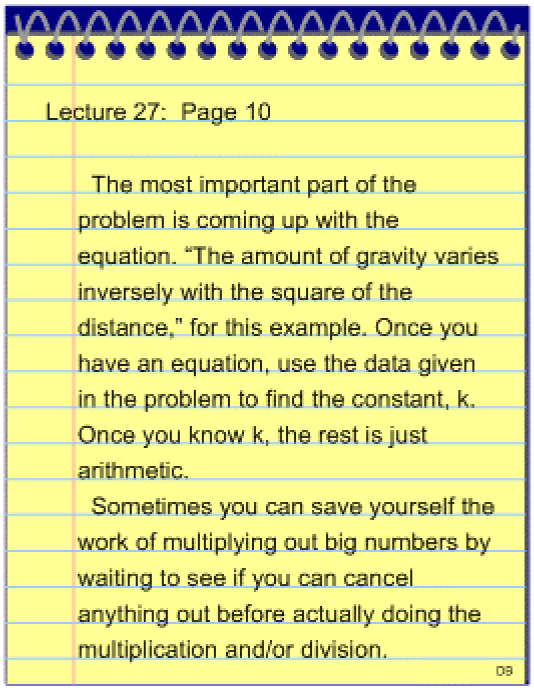
$$= \frac{81 \cdot \overset{\overset{1}{\cancel{8}}}{\cancel{64}}}{{}_{3}\cancel{24}\ \ \cancel{24}_{3}} = \frac{81}{9} = 9$$

There would only be 9 pounds of force between two bodies separated by 24 feet.

Answer: e

D3

CEMP027-10



Lecture 27: Page 10

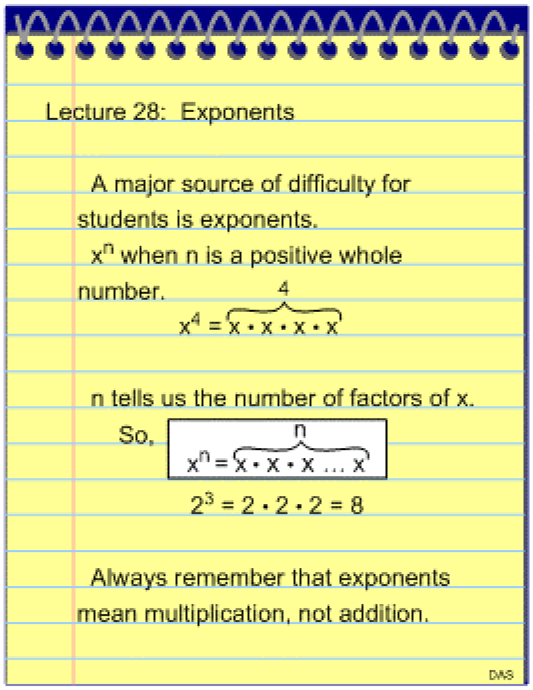
The most important part of the problem is coming up with the equation. "The amount of gravity varies inversely with the square of the distance," for this example. Once you have an equation, use the data given in the problem to find the constant, k. Once you know k, the rest is just arithmetic.

Sometimes you can save yourself the work of multiplying out big numbers by waiting to see if you can cancel anything out before actually doing the multiplication and/or division.

D3

Lecture 28 Notes

CEMP028-01



Lecture 28: Exponents

A major source of difficulty for students is exponents.

x^n when n is a positive whole number.

$$x^4 = \overbrace{x \cdot x \cdot x \cdot x}^{4}$$

n tells us the number of factors of x.

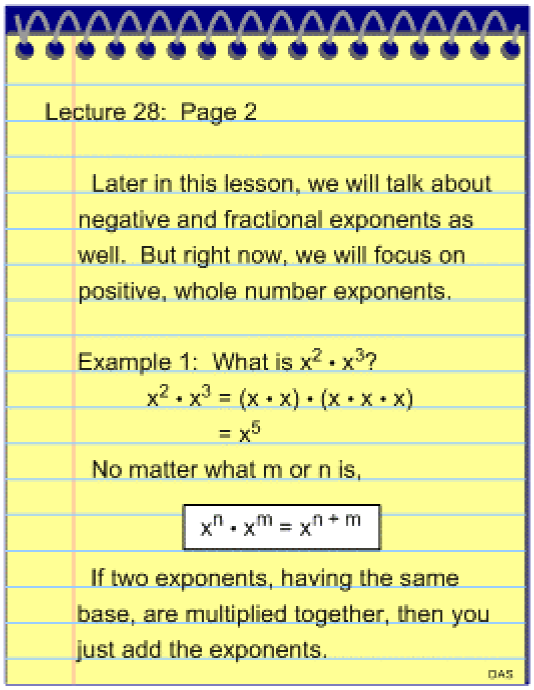
So, $\boxed{x^n = \overbrace{x \cdot x \cdot x \ldots x}^{n}}$

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

Always remember that exponents mean multiplication, not addition.

DAS

CEMP028-02



Lecture 28: Page 2

Later in this lesson, we will talk about negative and fractional exponents as well. But right now, we will focus on positive, whole number exponents.

Example 1: What is $x^2 \cdot x^3$?

$$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x)$$
$$= x^5$$

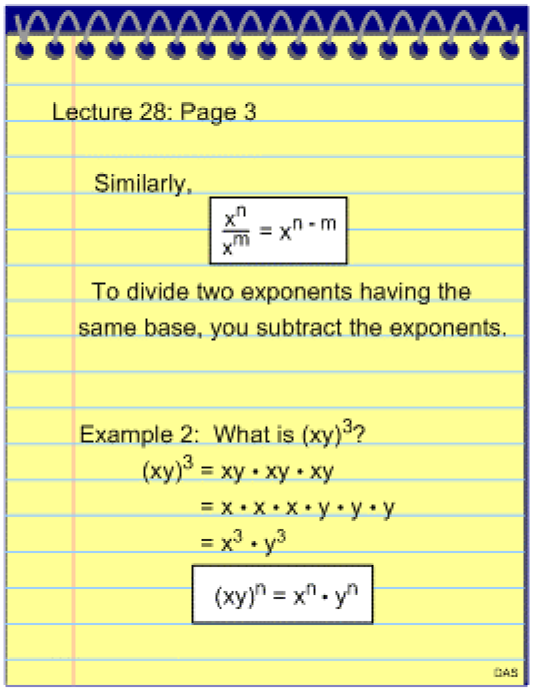
No matter what m or n is,

$$\boxed{x^n \cdot x^m = x^{n+m}}$$

If two exponents, having the same base, are multiplied together, then you just add the exponents.

DAS

CEMP028-03



Lecture 28: Page 3

Similarly,

$$\boxed{\frac{x^n}{x^m} = x^{n-m}}$$

To divide two exponents having the same base, you subtract the exponents.

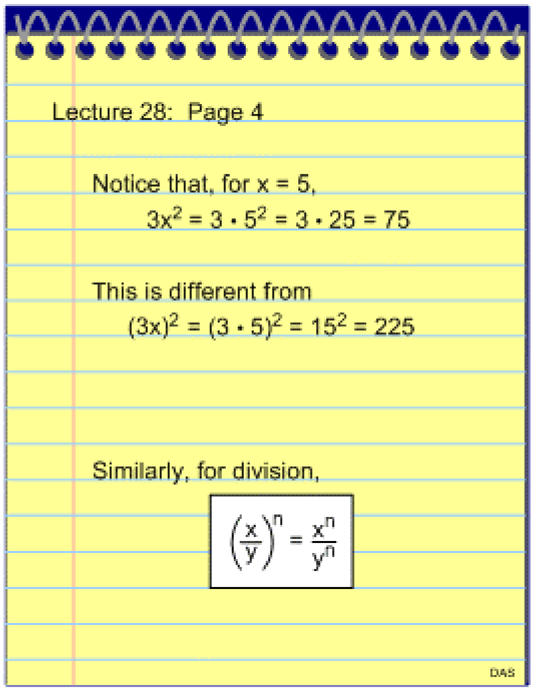
Example 2: What is $(xy)^3$?

$$(xy)^3 = xy \cdot xy \cdot xy$$
$$= x \cdot x \cdot x \cdot y \cdot y \cdot y$$
$$= x^3 \cdot y^3$$

$$\boxed{(xy)^n = x^n \cdot y^n}$$

DAS

CEMP028-04



Lecture 28: Page 4

Notice that, for x = 5,

$$3x^2 = 3 \cdot 5^2 = 3 \cdot 25 = 75$$

This is different from

$$(3x)^2 = (3 \cdot 5)^2 = 15^2 = 225$$

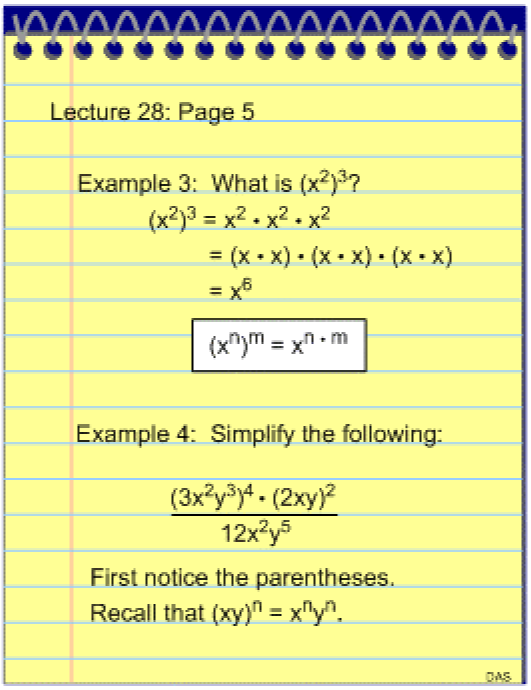
Similarly, for division,

$$\boxed{\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}}$$

DAS

Lecture 28 Notes, Continued

CEMP028-05



Lecture 28: Page 5

Example 3: What is $(x^2)^3$?

$(x^2)^3 = x^2 \cdot x^2 \cdot x^2$

$= (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x)$

$= x^6$

$\boxed{(x^n)^m = x^{n \cdot m}}$

Example 4: Simplify the following:

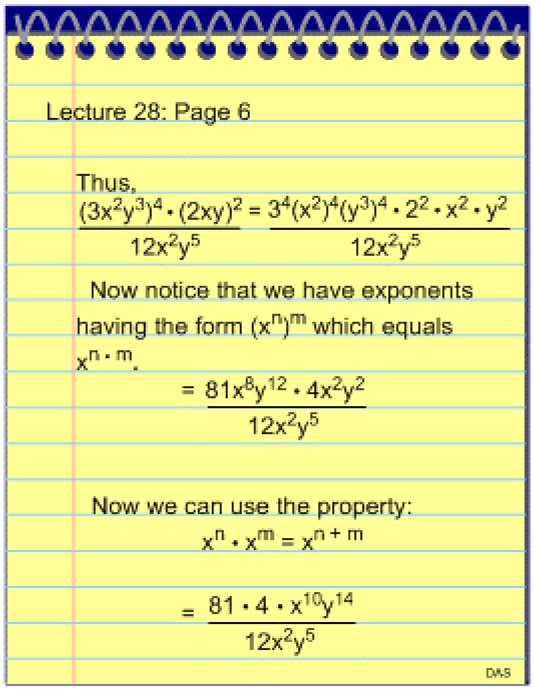
$$\frac{(3x^2y^3)^4 \cdot (2xy)^2}{12x^2y^5}$$

First notice the parentheses.

Recall that $(xy)^n = x^ny^n$.

DAS

CEMP028-06



Lecture 28: Page 6

Thus,

$$\frac{(3x^2y^3)^4 \cdot (2xy)^2}{12x^2y^5} = \frac{3^4(x^2)^4(y^3)^4 \cdot 2^2 \cdot x^2 \cdot y^2}{12x^2y^5}$$

Now notice that we have exponents having the form $(x^n)^m$ which equals $x^{n \cdot m}$.

$$= \frac{81x^8y^{12} \cdot 4x^2y^2}{12x^2y^5}$$

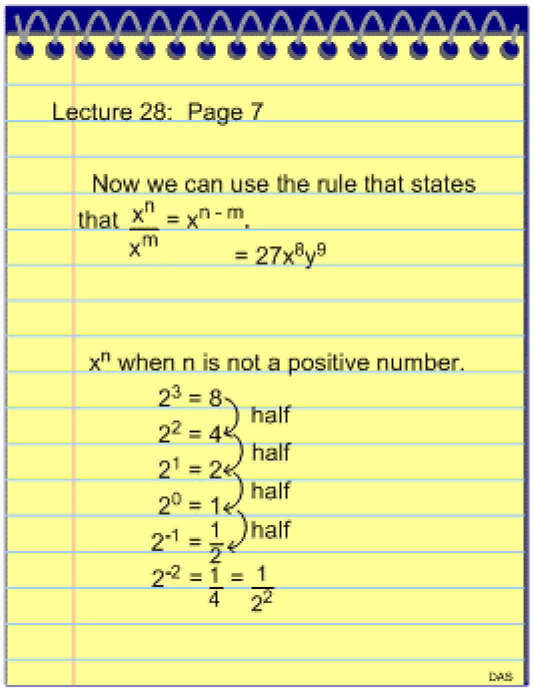
Now we can use the property:

$x^n \cdot x^m = x^{n+m}$

$$= \frac{81 \cdot 4 \cdot x^{10}y^{14}}{12x^2y^5}$$

DAS

CEMP028-07



Lecture 28: Page 7

Now we can use the rule that states that $\frac{x^n}{x^m} = x^{n-m}$.

$= 27x^8y^9$

x^n when n is not a positive number.

$2^3 = 8$

half

$2^2 = 4$

half

$2^1 = 2$

half

$2^0 = 1$

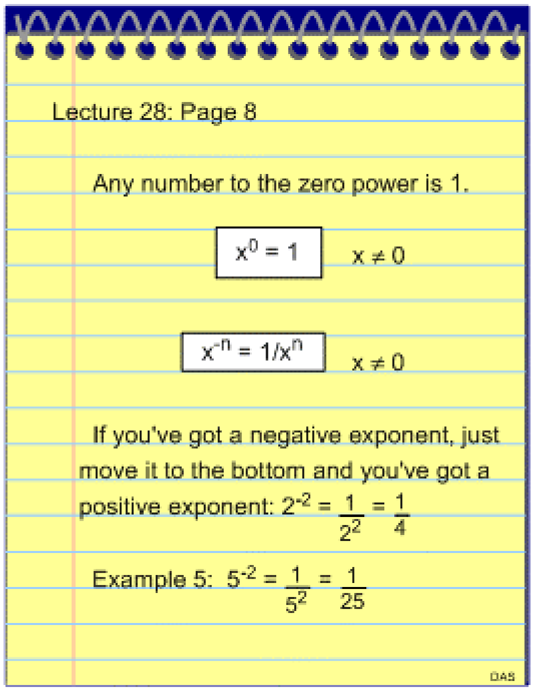
half

$2^{-1} = \frac{1}{2}$

$2^{-2} = \frac{1}{4} = \frac{1}{2^2}$

DAS

CEMP028-08



Lecture 28: Page 8

Any number to the zero power is 1.

$\boxed{x^0 = 1}$ $x \neq 0$

$\boxed{x^{-n} = 1/x^n}$ $x \neq 0$

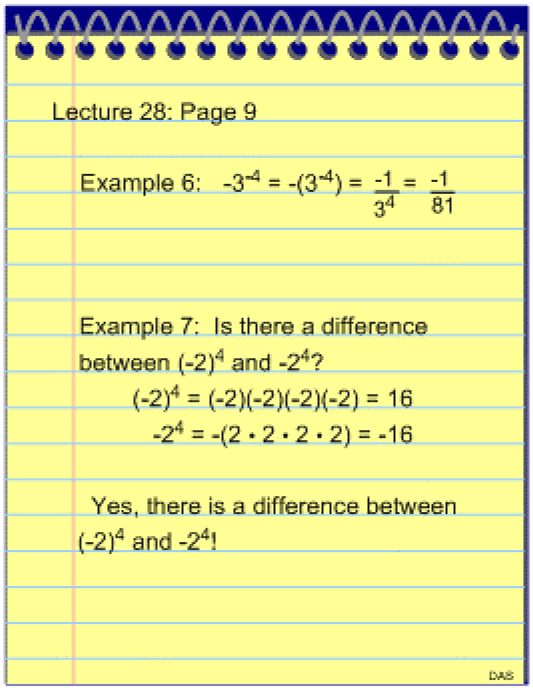
If you've got a negative exponent, just move it to the bottom and you've got a positive exponent: $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

Example 5: $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

DAS

Lecture 28 Notes, Continued

CEMP028-09



Lecture 28: Page 9

Example 6: $-3^{-4} = -(3^{-4}) = \frac{-1}{3^4} = \frac{-1}{81}$

Example 7: Is there a difference between $(-2)^4$ and -2^4?

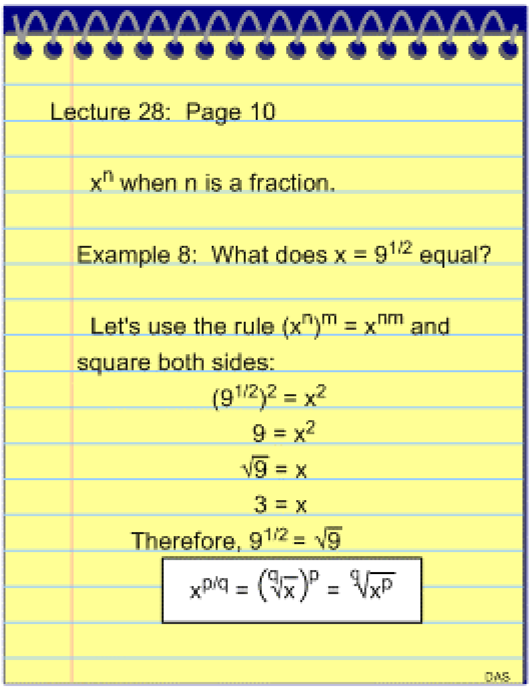
$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

$$-2^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$$

Yes, there is a difference between $(-2)^4$ and -2^4!

DAS

CEMP028-10



Lecture 28: Page 10

x^n when n is a fraction.

Example 8: What does $x = 9^{1/2}$ equal?

Let's use the rule $(x^n)^m = x^{nm}$ and square both sides:

$$(9^{1/2})^2 = x^2$$

$$9 = x^2$$

$$\sqrt{9} = x$$

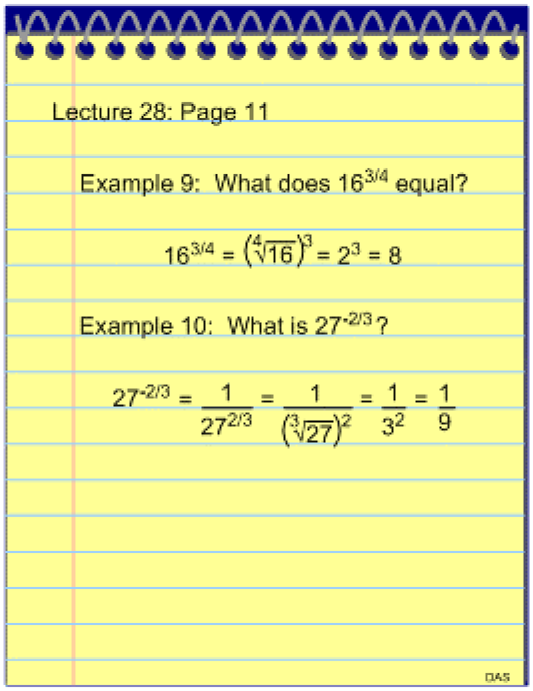
$$3 = x$$

Therefore, $9^{1/2} = \sqrt{9}$

$$\boxed{x^{p/q} = \left(\sqrt[q]{x}\right)^p = \sqrt[q]{x^p}}$$

DAS

CEMP028-11



Lecture 28: Page 11

Example 9: What does $16^{3/4}$ equal?

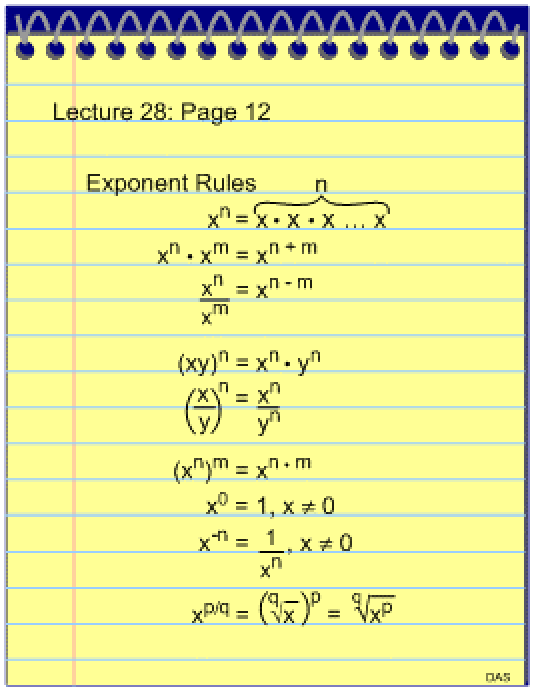
$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$$

Example 10: What is $27^{-2/3}$?

$$27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{\left(\sqrt[3]{27}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$$

DAS

CEMP028-12



Lecture 28: Page 12

Exponent Rules

$$x^n = \overbrace{x \cdot x \cdot x \ldots x}^{n}$$

$$x^n \cdot x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m}$$

$$(xy)^n = x^n \cdot y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$(x^n)^m = x^{n \cdot m}$$

$$x^0 = 1,\ x \neq 0$$

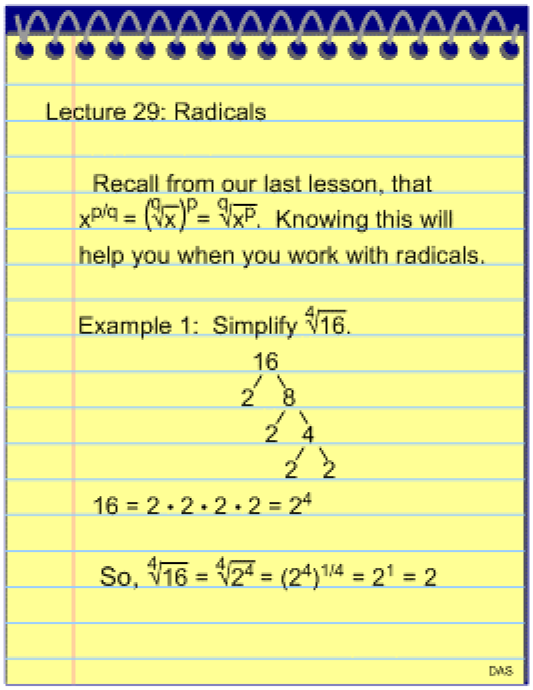
$$x^{-n} = \frac{1}{x^n},\ x \neq 0$$

$$x^{p/q} = \left(\sqrt[q]{x}\right)^p = \sqrt[q]{x^p}$$

DAS

Lecture 29 Notes

CEMP029-01



Lecture 29: Radicals

Recall from our last lesson, that $x^{p/q} = (\sqrt[q]{x})^p = \sqrt[q]{x^p}$. Knowing this will help you when you work with radicals.

Example 1: Simplify $\sqrt[4]{16}$.

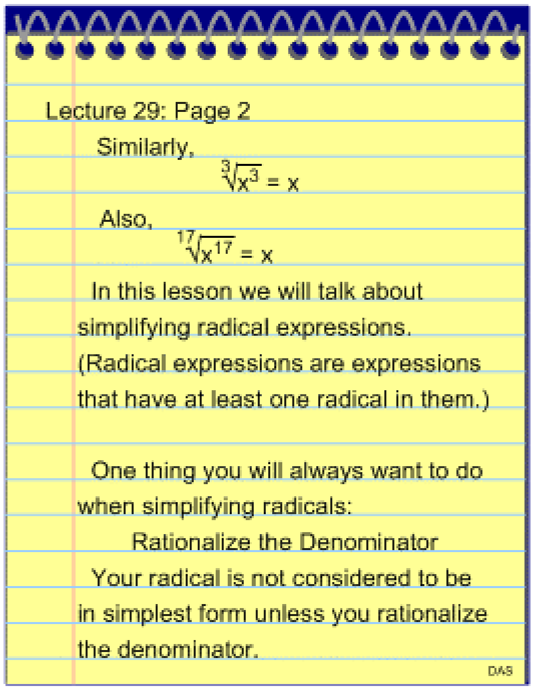
16
2 8
2 4
2 2

$16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$

So, $\sqrt[4]{16} = \sqrt[4]{2^4} = (2^4)^{1/4} = 2^1 = 2$

DAS

CEMP029-02



Lecture 29: Page 2

Similarly,

$$\sqrt[3]{x^3} = x$$

Also,

$$\sqrt[17]{x^{17}} = x$$

In this lesson we will talk about simplifying radical expressions. (Radical expressions are expressions that have at least one radical in them.)

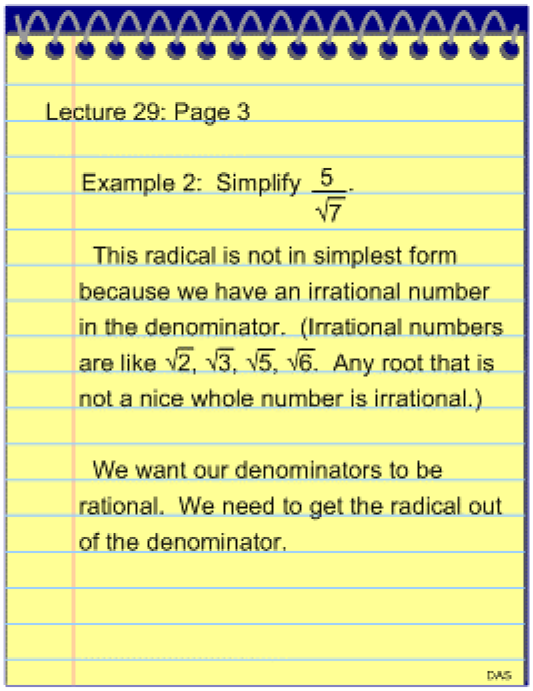
One thing you will always want to do when simplifying radicals:

Rationalize the Denominator

Your radical is not considered to be in simplest form unless you rationalize the denominator.

DAS

CEMP029-03



Lecture 29: Page 3

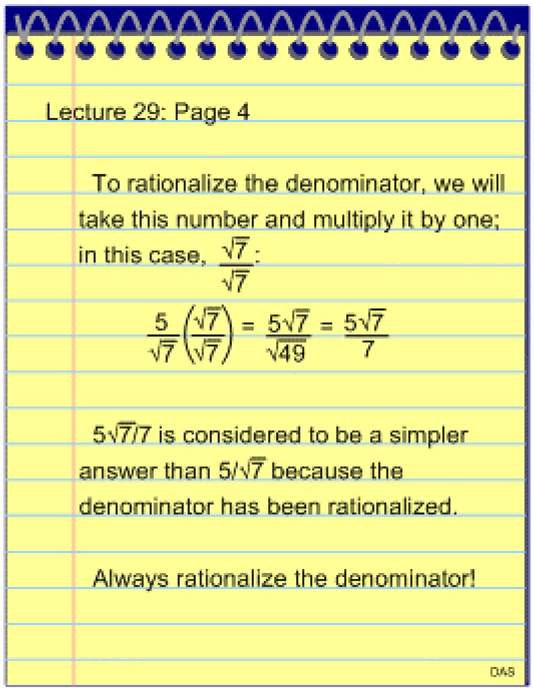
Example 2: Simplify $\frac{5}{\sqrt{7}}$.

This radical is not in simplest form because we have an irrational number in the denominator. (Irrational numbers are like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$. Any root that is not a nice whole number is irrational.)

We want our denominators to be rational. We need to get the radical out of the denominator.

DAS

CEMP029-04



Lecture 29: Page 4

To rationalize the denominator, we will take this number and multiply it by one; in this case, $\frac{\sqrt{7}}{\sqrt{7}}$:

$$\frac{5}{\sqrt{7}}\left(\frac{\sqrt{7}}{\sqrt{7}}\right) = \frac{5\sqrt{7}}{\sqrt{49}} = \frac{5\sqrt{7}}{7}$$

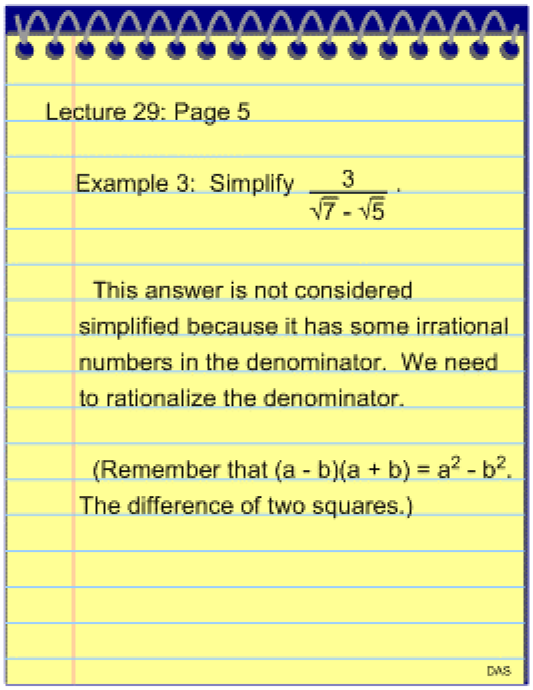
$5\sqrt{7}/7$ is considered to be a simpler answer than $5/\sqrt{7}$ because the denominator has been rationalized.

Always rationalize the denominator!

DAS

Lecture 29 Notes, Continued

CEMP029-05



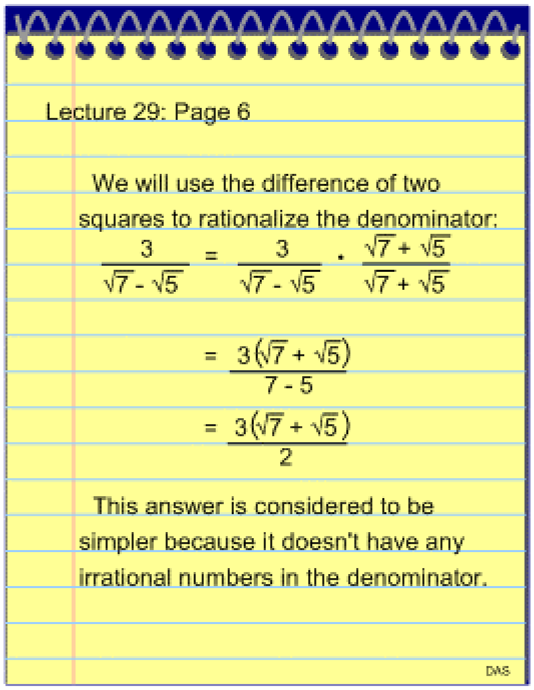
Lecture 29: Page 5

Example 3: Simplify $\frac{3}{\sqrt{7} - \sqrt{5}}$.

This answer is not considered simplified because it has some irrational numbers in the denominator. We need to rationalize the denominator.

(Remember that $(a - b)(a + b) = a^2 - b^2$. The difference of two squares.)

CEMP029-06



Lecture 29: Page 6

We will use the difference of two squares to rationalize the denominator:

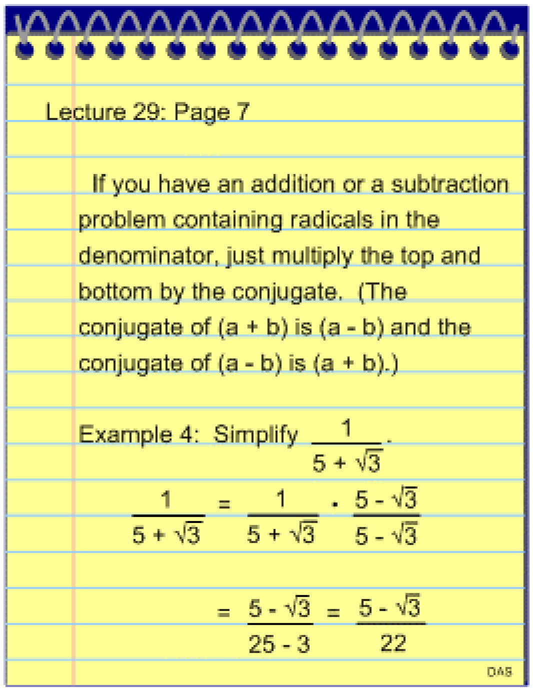
$$\frac{3}{\sqrt{7} - \sqrt{5}} = \frac{3}{\sqrt{7} - \sqrt{5}} \cdot \frac{\sqrt{7} + \sqrt{5}}{\sqrt{7} + \sqrt{5}}$$

$$= \frac{3\left(\sqrt{7} + \sqrt{5}\right)}{7 - 5}$$

$$= \frac{3\left(\sqrt{7} + \sqrt{5}\right)}{2}$$

This answer is considered to be simpler because it doesn't have any irrational numbers in the denominator.

CEMP029-07



Lecture 29: Page 7

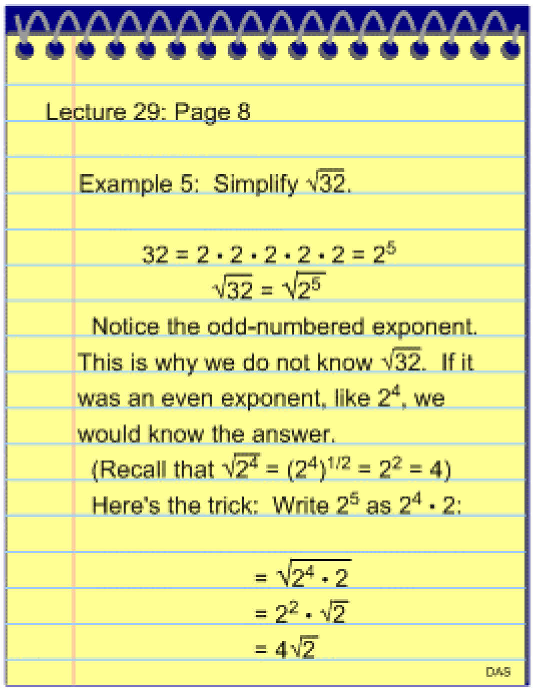
If you have an addition or a subtraction problem containing radicals in the denominator, just multiply the top and bottom by the conjugate. (The conjugate of (a + b) is (a - b) and the conjugate of (a - b) is (a + b).)

Example 4: Simplify $\frac{1}{5 + \sqrt{3}}$.

$$\frac{1}{5 + \sqrt{3}} = \frac{1}{5 + \sqrt{3}} \cdot \frac{5 - \sqrt{3}}{5 - \sqrt{3}}$$

$$= \frac{5 - \sqrt{3}}{25 - 3} = \frac{5 - \sqrt{3}}{22}$$

CEMP029-08



Lecture 29: Page 8

Example 5: Simplify $\sqrt{32}$.

$$32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$$
$$\sqrt{32} = \sqrt{2^5}$$

Notice the odd-numbered exponent. This is why we do not know $\sqrt{32}$. If it was an even exponent, like 2^4, we would know the answer.

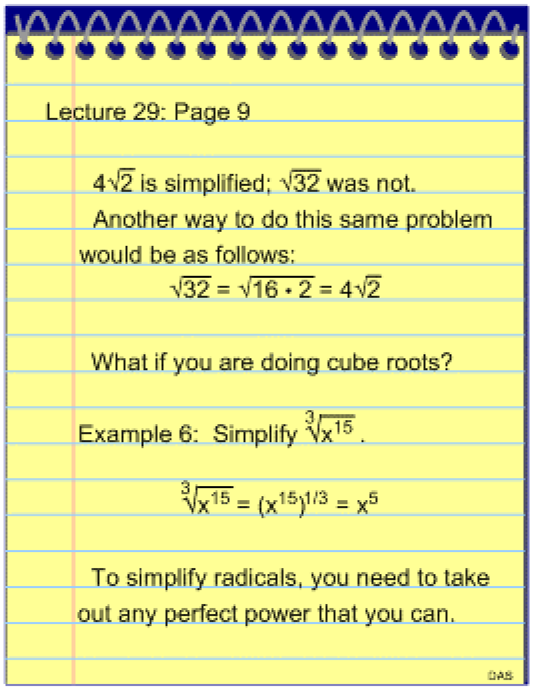
(Recall that $\sqrt{2^4} = (2^4)^{1/2} = 2^2 = 4$)

Here's the trick: Write 2^5 as $2^4 \cdot 2$:

$$= \sqrt{2^4 \cdot 2}$$
$$= 2^2 \cdot \sqrt{2}$$
$$= 4\sqrt{2}$$

Lecture 29 Notes, Continued

CEMP029-09



Lecture 29: Page 9

$4\sqrt{2}$ is simplified; $\sqrt{32}$ was not.

Another way to do this same problem would be as follows:

$$\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

What if you are doing cube roots?

Example 6: Simplify $\sqrt[3]{x^{15}}$.

$$\sqrt[3]{x^{15}} = (x^{15})^{1/3} = x^5$$

To simplify radicals, you need to take out any perfect power that you can.

DAS

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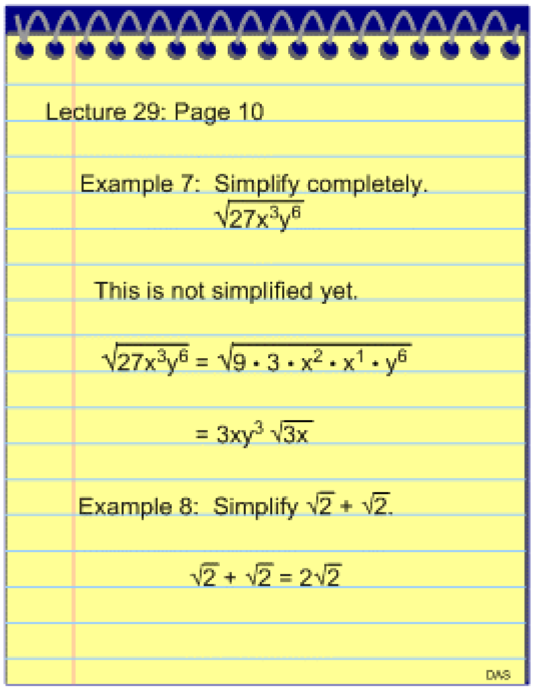


Lecture 29: Page 10

Example 7: Simplify completely.

$$\sqrt{27x^3y^6}$$

This is not simplified yet.

$$\sqrt{27x^3y^6} = \sqrt{9 \cdot 3 \cdot x^2 \cdot x^1 \cdot y^6}$$

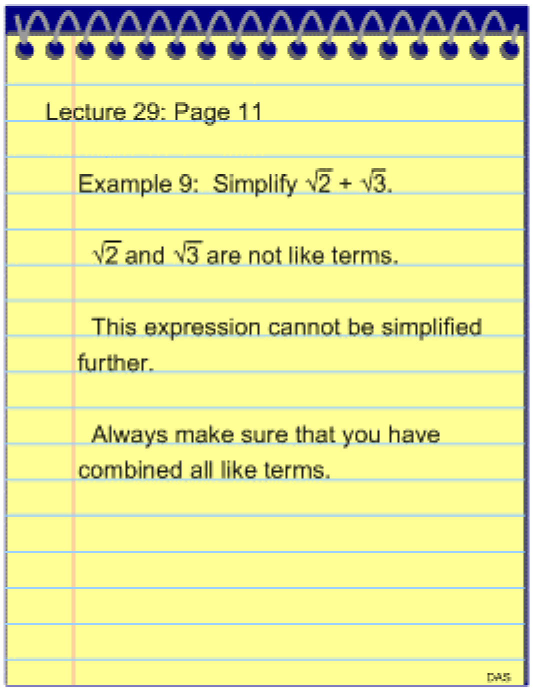
$$= 3xy^3\sqrt{3x}$$

Example 8: Simplify $\sqrt{2} + \sqrt{2}$.

$$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

DAS

CEMP029-11



Lecture 29: Page 11

Example 9: Simplify $\sqrt{2} + \sqrt{3}$.

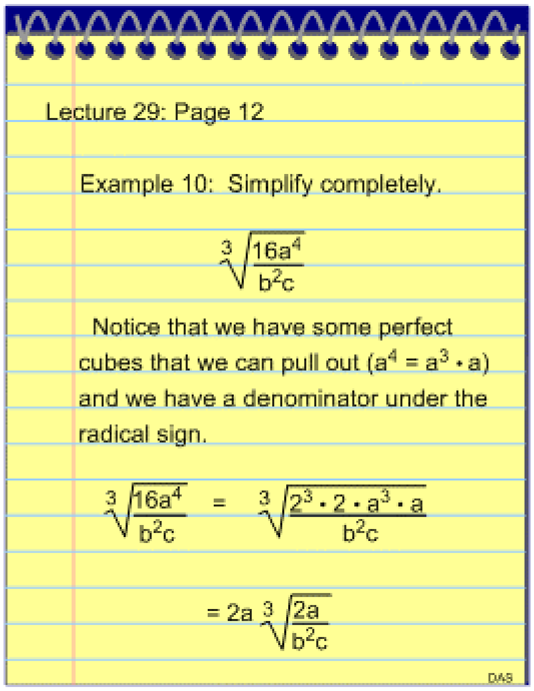
$\sqrt{2}$ and $\sqrt{3}$ are not like terms.

This expression cannot be simplified further.

Always make sure that you have combined all like terms.

DAS

CEMP029-12



Lecture 29: Page 12

Example 10: Simplify completely.

$$\sqrt[3]{\frac{16a^4}{b^2c}}$$

Notice that we have some perfect cubes that we can pull out ($a^4 = a^3 \cdot a$) and we have a denominator under the radical sign.

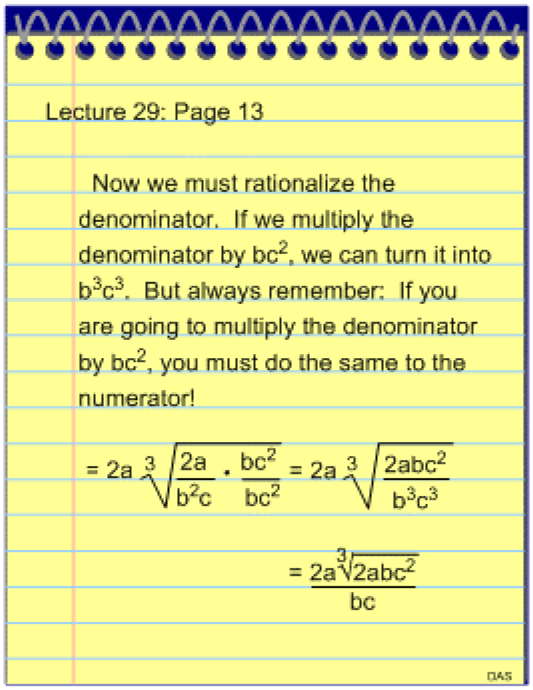
$$\sqrt[3]{\frac{16a^4}{b^2c}} = \sqrt[3]{\frac{2^3 \cdot 2 \cdot a^3 \cdot a}{b^2c}}$$

$$= 2a\sqrt[3]{\frac{2a}{b^2c}}$$

DAS

Lecture 29 Notes, Continued

CEMP029-13



Lecture 29: Page 13

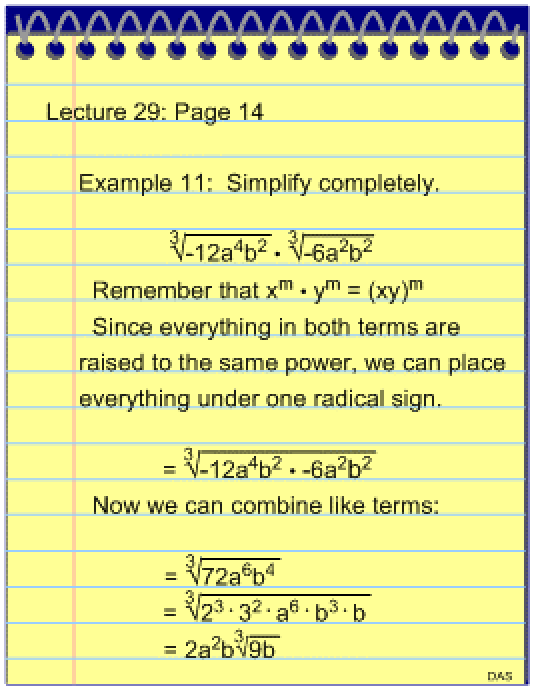
Now we must rationalize the denominator. If we multiply the denominator by bc^2, we can turn it into b^3c^3. But always remember: If you are going to multiply the denominator by bc^2, you must do the same to the numerator!

$$= 2a\sqrt[3]{\frac{2a}{b^2c}\cdot\frac{bc^2}{bc^2}} = 2a\sqrt[3]{\frac{2abc^2}{b^3c^3}}$$

$$= \frac{2a\sqrt[3]{2abc^2}}{bc}$$

DAS

CEMP029-14



Lecture 29: Page 14

Example 11: Simplify completely.

$$\sqrt[3]{-12a^4b^2}\cdot\sqrt[3]{-6a^2b^2}$$

Remember that $x^m \cdot y^m = (xy)^m$

Since everything in both terms are raised to the same power, we can place everything under one radical sign.

$$= \sqrt[3]{-12a^4b^2\cdot -6a^2b^2}$$

Now we can combine like terms:

$$= \sqrt[3]{72a^6b^4}$$

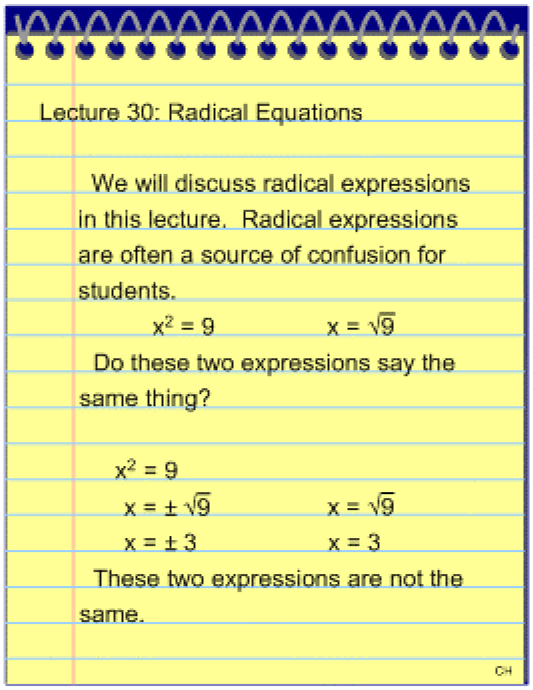
$$= \sqrt[3]{2^3\cdot 3^2\cdot a^6\cdot b^3\cdot b}$$

$$= 2a^2b\sqrt[3]{9b}$$

DAS

Lecture 30 Notes

CEMP030-01



Lecture 30: Radical Equations

We will discuss radical expressions in this lecture. Radical expressions are often a source of confusion for students.

$x^2 = 9$ $\quad\quad$ $x = \sqrt{9}$

Do these two expressions say the same thing?

$x^2 = 9$

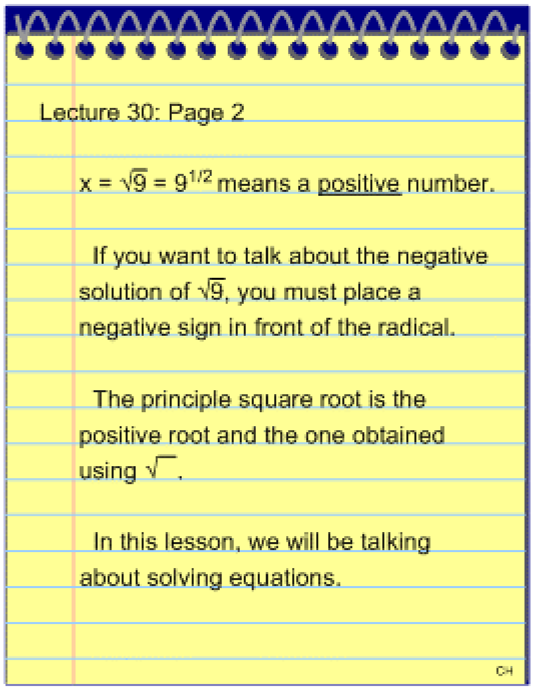
$x = \pm\sqrt{9}$ $\quad\quad$ $x = \sqrt{9}$

$x = \pm 3$ $\quad\quad$ $x = 3$

These two expressions are not the same.

CH

CEMP030-02



Lecture 30: Page 2

$x = \sqrt{9} = 9^{1/2}$ means a <u>positive</u> number.

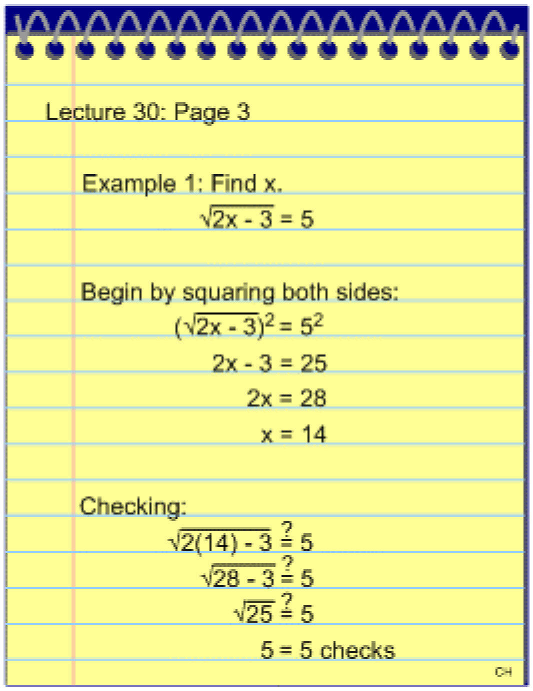
If you want to talk about the negative solution of $\sqrt{9}$, you must place a negative sign in front of the radical.

The principle square root is the positive root and the one obtained using $\sqrt{}$.

In this lesson, we will be talking about solving equations.

CH

CEMP030-03



Lecture 30: Page 3

Example 1: Find x.

$\sqrt{2x - 3} = 5$

Begin by squaring both sides:

$(\sqrt{2x - 3})^2 = 5^2$

$2x - 3 = 25$

$2x = 28$

$x = 14$

Checking:

$\sqrt{2(14) - 3} \overset{?}{=} 5$

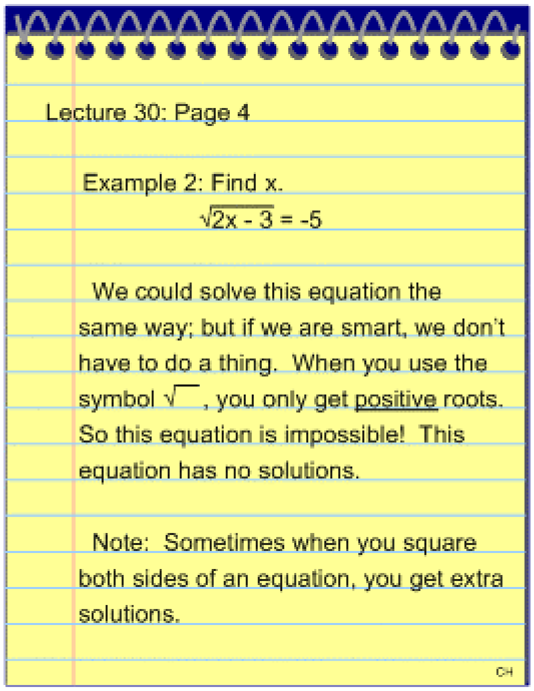
$\sqrt{28 - 3} \overset{?}{=} 5$

$\sqrt{25} \overset{?}{=} 5$

$5 = 5$ checks

CH

CEMP030-04



Lecture 30: Page 4

Example 2: Find x.

$\sqrt{2x - 3} = -5$

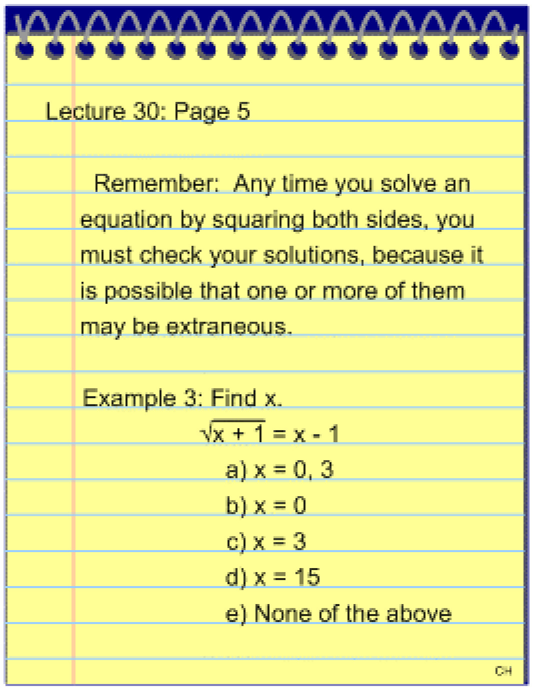
We could solve this equation the same way; but if we are smart, we don't have to do a thing. When you use the symbol $\sqrt{}$, you only get <u>positive</u> roots. So this equation is impossible! This equation has no solutions.

Note: Sometimes when you square both sides of an equation, you get extra solutions.

CH

Lecture 30 Notes, Continued

CEMP030-05



Lecture 30: Page 5

Remember: Any time you solve an equation by squaring both sides, you must check your solutions, because it is possible that one or more of them may be extraneous.

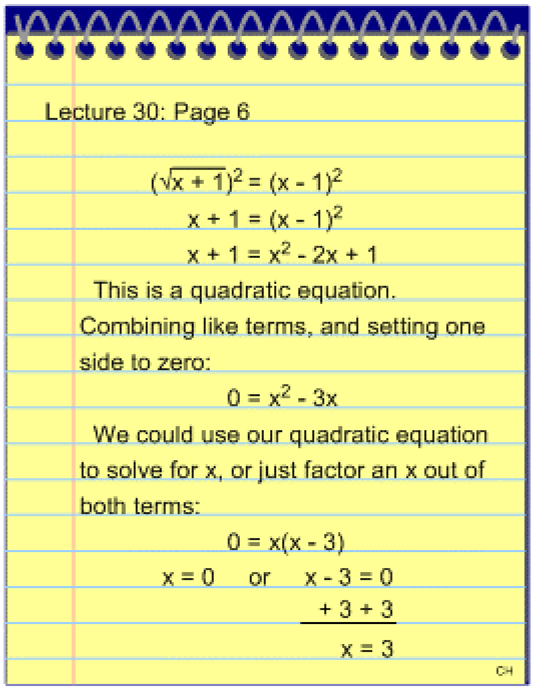
Example 3: Find x.

$$\sqrt{x+1} = x - 1$$

a) $x = 0, 3$
b) $x = 0$
c) $x = 3$
d) $x = 15$
e) None of the above

CH

CEMP030-06



Lecture 30: Page 6

$$(\sqrt{x+1})^2 = (x - 1)^2$$
$$x + 1 = (x - 1)^2$$
$$x + 1 = x^2 - 2x + 1$$

This is a quadratic equation. Combining like terms, and setting one side to zero:

$$0 = x^2 - 3x$$

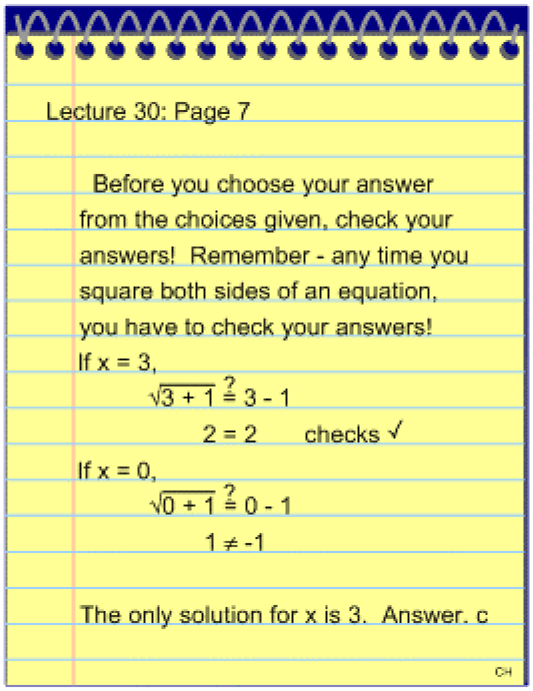
We could use our quadratic equation to solve for x, or just factor an x out of both terms:

$$0 = x(x - 3)$$

$$x = 0 \quad \text{or} \quad \begin{array}{r} x - 3 = 0 \\ \underline{+3 + 3} \\ x = 3 \end{array}$$

CH

CEMP030-07



Lecture 30: Page 7

Before you choose your answer from the choices given, check your answers! Remember - any time you square both sides of an equation, you have to check your answers!

If $x = 3$,

$$\sqrt{3+1} \overset{?}{=} 3 - 1$$
$$2 = 2 \qquad \text{checks } \checkmark$$

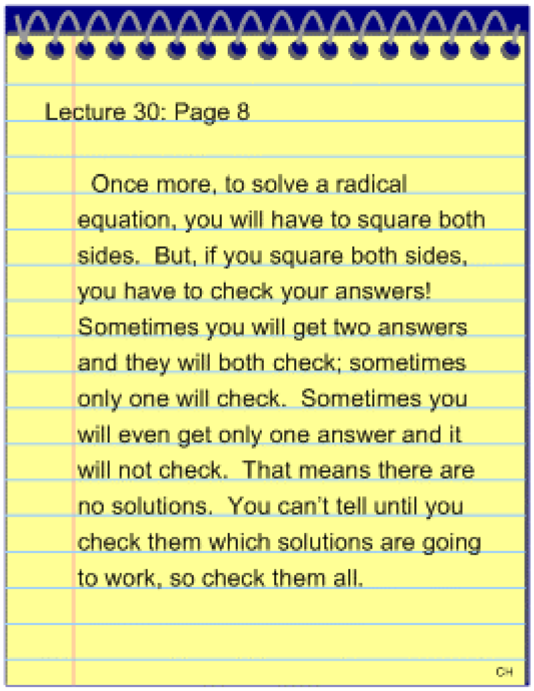
If $x = 0$,

$$\sqrt{0+1} \overset{?}{=} 0 - 1$$
$$1 \neq -1$$

The only solution for x is 3. Answer. c

CH

CEMP030-08



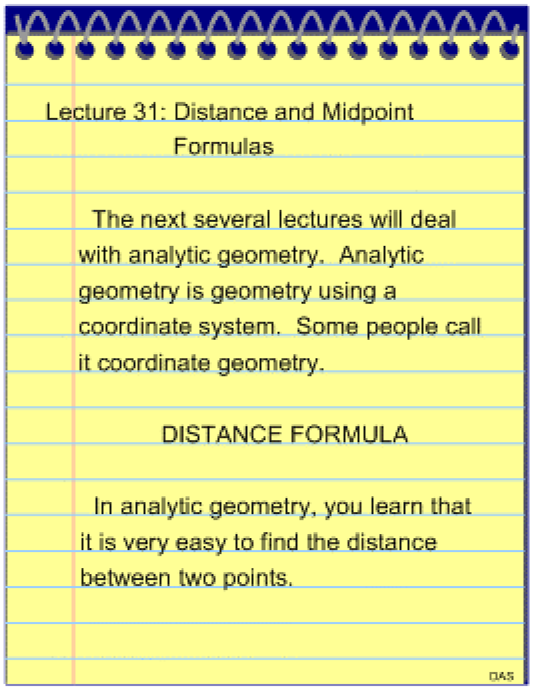
Lecture 30: Page 8

Once more, to solve a radical equation, you will have to square both sides. But, if you square both sides, you have to check your answers! Sometimes you will get two answers and they will both check; sometimes only one will check. Sometimes you will even get only one answer and it will not check. That means there are no solutions. You can't tell until you check them which solutions are going to work, so check them all.

CH

Lecture 31 Notes

CEMP031-01



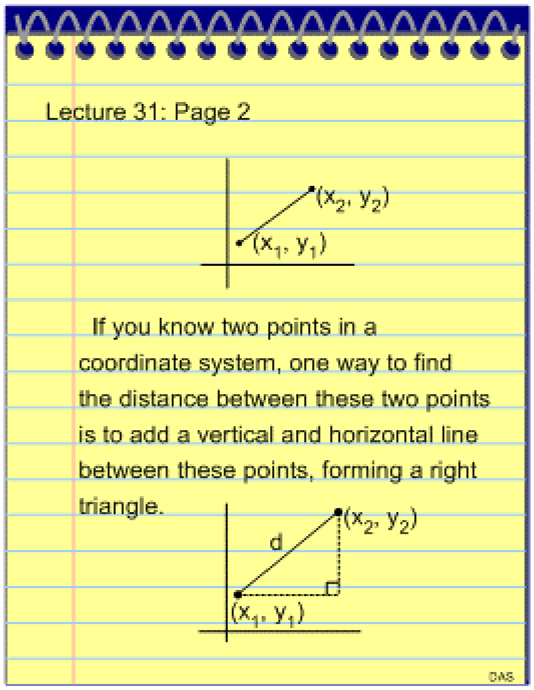
Lecture 31: Distance and Midpoint Formulas

The next several lectures will deal with analytic geometry. Analytic geometry is geometry using a coordinate system. Some people call it coordinate geometry.

DISTANCE FORMULA

In analytic geometry, you learn that it is very easy to find the distance between two points.

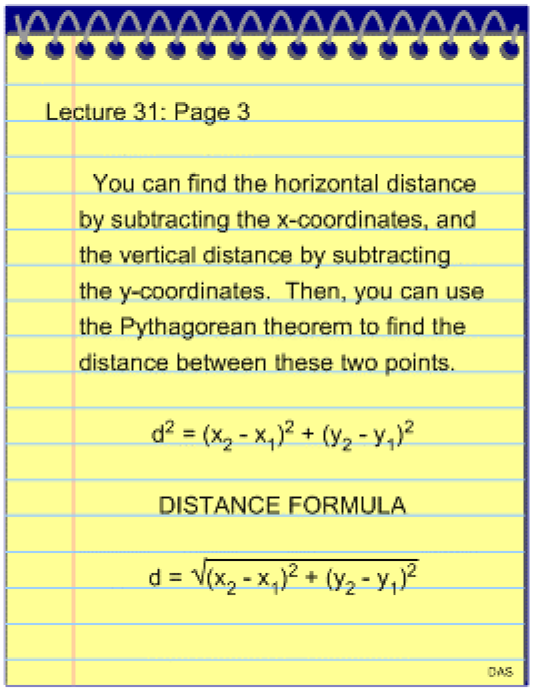
CEMP031-02



Lecture 31: Page 2

If you know two points in a coordinate system, one way to find the distance between these two points is to add a vertical and horizontal line between these points, forming a right triangle.

CEMP031-03



Lecture 31: Page 3

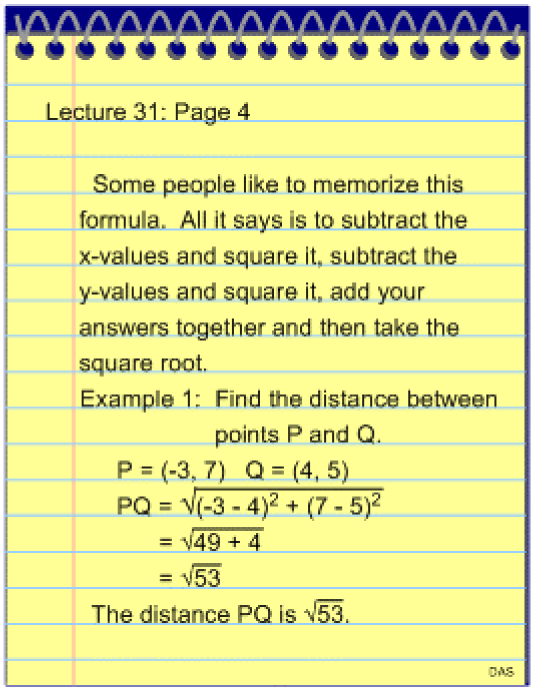
You can find the horizontal distance by subtracting the x-coordinates, and the vertical distance by subtracting the y-coordinates. Then, you can use the Pythagorean theorem to find the distance between these two points.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

DISTANCE FORMULA

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

CEMP031-04



Lecture 31: Page 4

Some people like to memorize this formula. All it says is to subtract the x-values and square it, subtract the y-values and square it, add your answers together and then take the square root.

Example 1: Find the distance between points P and Q.

$P = (-3, 7) \quad Q = (4, 5)$

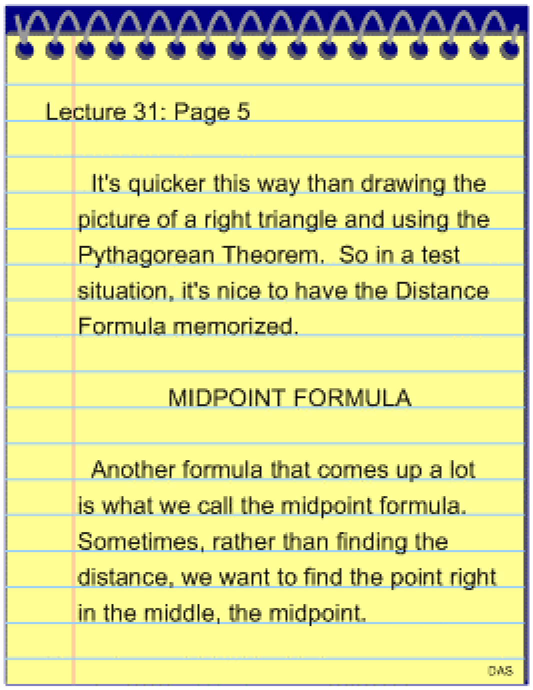
$PQ = \sqrt{(-3 - 4)^2 + (7 - 5)^2}$

$= \sqrt{49 + 4}$

$= \sqrt{53}$

The distance PQ is $\sqrt{53}$.

CEMP031-05



Lecture 31: Page 5

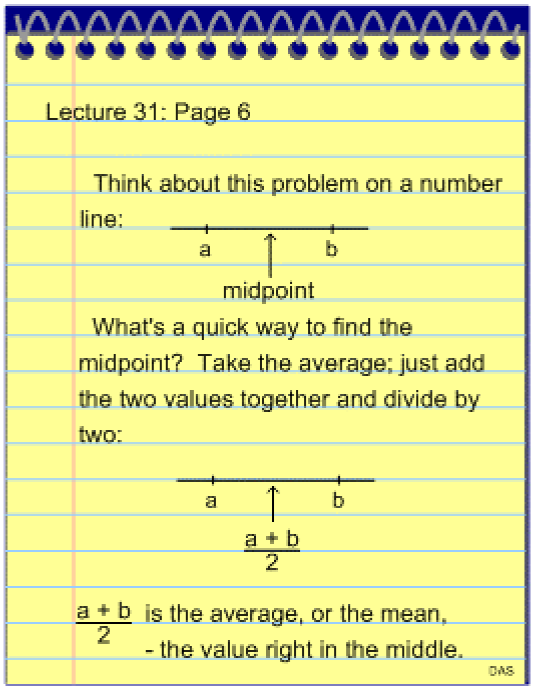
It's quicker this way than drawing the picture of a right triangle and using the Pythagorean Theorem. So in a test situation, it's nice to have the Distance Formula memorized.

MIDPOINT FORMULA

Another formula that comes up a lot is what we call the midpoint formula. Sometimes, rather than finding the distance, we want to find the point right in the middle, the midpoint.

DAS

CEMP031-06



Lecture 31: Page 6

Think about this problem on a number line:

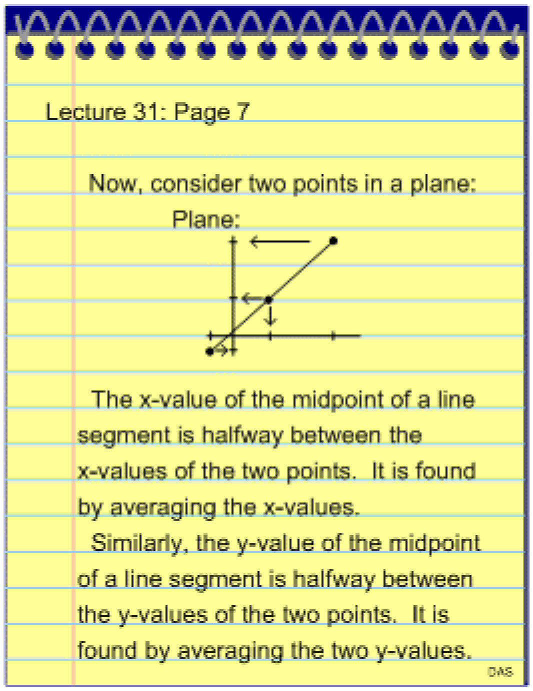
What's a quick way to find the midpoint? Take the average; just add the two values together and divide by two:

$\frac{a+b}{2}$

$\frac{a+b}{2}$ is the average, or the mean, - the value right in the middle.

DAS

CEMP031-07



Lecture 31: Page 7

Now, consider two points in a plane:

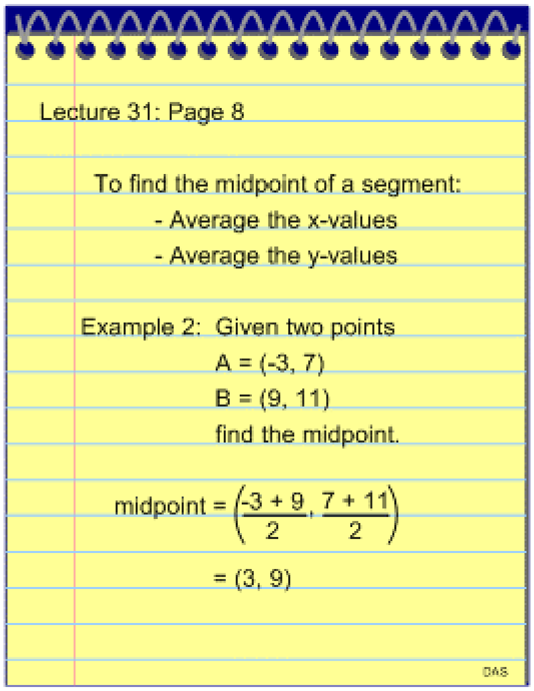
Plane:

The x-value of the midpoint of a line segment is halfway between the x-values of the two points. It is found by averaging the x-values.

Similarly, the y-value of the midpoint of a line segment is halfway between the y-values of the two points. It is found by averaging the two y-values.

DAS

CEMP031-08



Lecture 31: Page 8

To find the midpoint of a segment:

- Average the x-values
- Average the y-values

Example 2: Given two points

A = (-3, 7)

B = (9, 11)

find the midpoint.

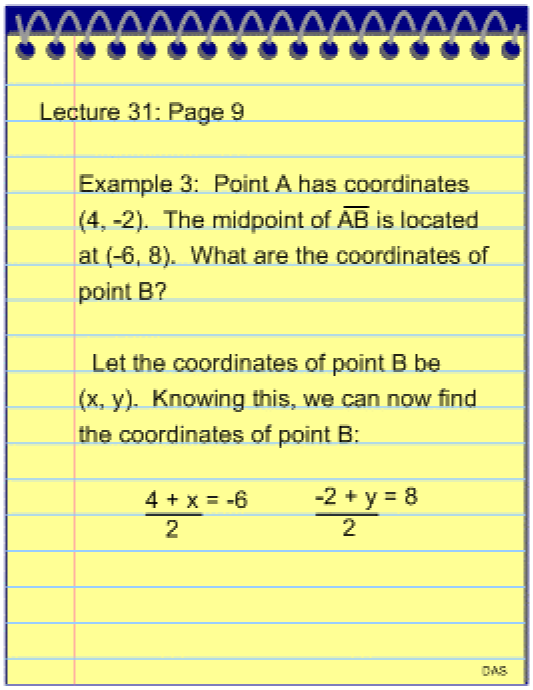
$$\text{midpoint} = \left(\frac{-3+9}{2}, \frac{7+11}{2}\right)$$

$$= (3, 9)$$

DAS

Lecture 31 Notes, Continued

CEMP031-09



Lecture 31: Page 9

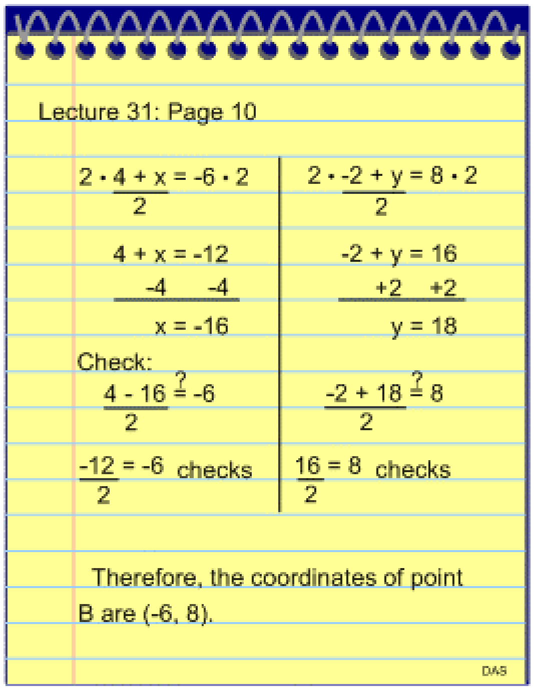
Example 3: Point A has coordinates (4, -2). The midpoint of $\overline{AB}$ is located at (-6, 8). What are the coordinates of point B?

Let the coordinates of point B be (x, y). Knowing this, we can now find the coordinates of point B:

$$\frac{4 + x}{2} = -6 \qquad \frac{-2 + y}{2} = 8$$

DAS

CEMP031-10



Lecture 31: Page 10

$$2 \cdot \frac{4 + x}{2} = -6 \cdot 2$$

$$4 + x = -12$$

$$\underline{\quad -4 \qquad -4}$$

$$x = -16$$

Check:

$$\frac{4 - 16}{2} \overset{?}{=} -6$$

$$\frac{-12}{2} = -6 \text{ checks}$$

$$2 \cdot \frac{-2 + y}{2} = 8 \cdot 2$$

$$-2 + y = 16$$

$$\underline{\quad +2 \qquad +2}$$

$$y = 18$$

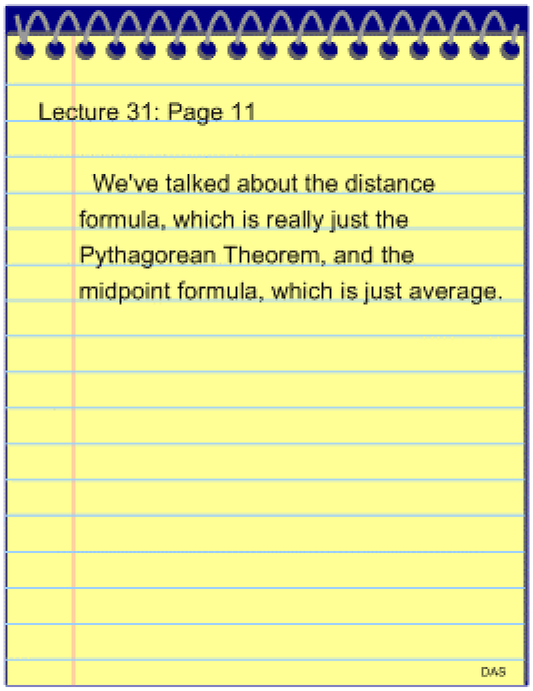
$$\frac{-2 + 18}{2} \overset{?}{=} 8$$

$$\frac{16}{2} = 8 \text{ checks}$$

Therefore, the coordinates of point B are (-6, 8).

DAS

CEMP031-11



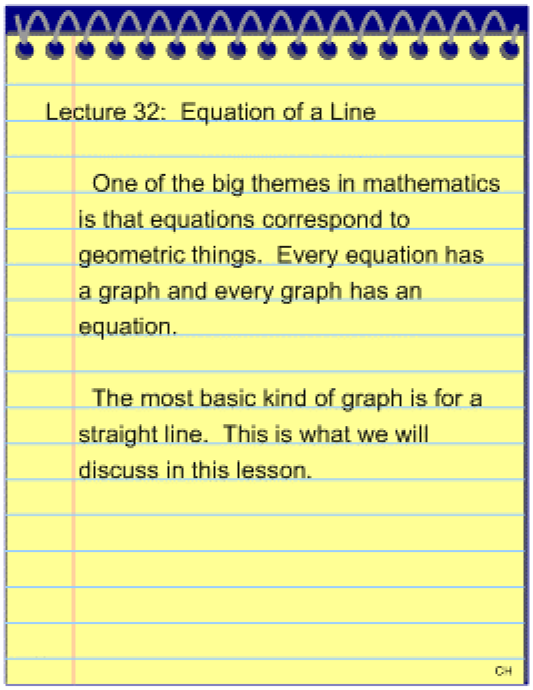
Lecture 31: Page 11

We've talked about the distance formula, which is really just the Pythagorean Theorem, and the midpoint formula, which is just average.

DAS

Lecture 32 Notes

CEMP032-01



Lecture 32: Equation of a Line

One of the big themes in mathematics is that equations correspond to geometric things. Every equation has a graph and every graph has an equation.

The most basic kind of graph is for a straight line. This is what we will discuss in this lesson.

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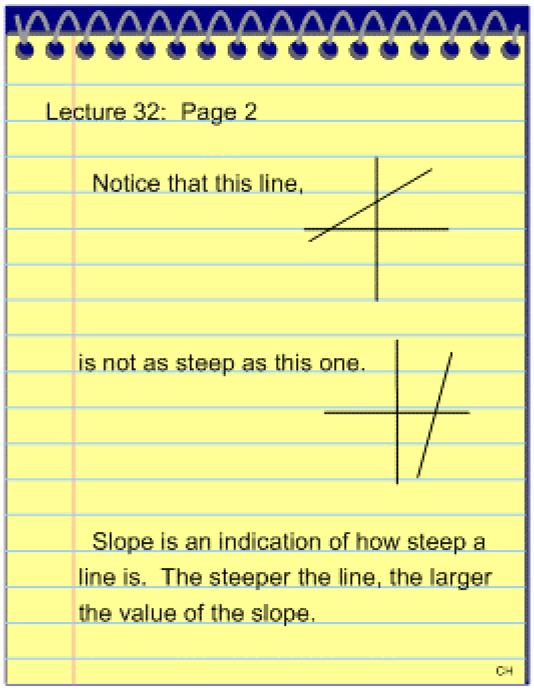


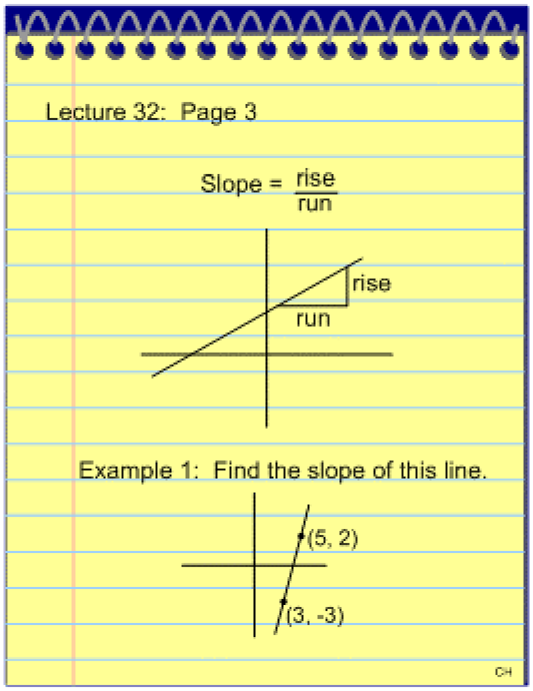
Lecture 32: Page 2

Notice that this line,

is not as steep as this one.

Slope is an indication of how steep a line is. The steeper the line, the larger the value of the slope.

CEMP032-03

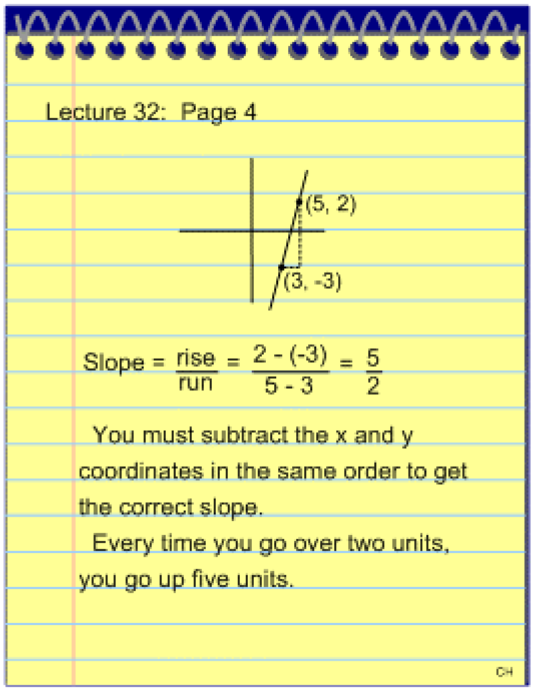


Lecture 32: Page 3

$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

Example 1: Find the slope of this line.

CEMP032-04



Lecture 32: Page 4

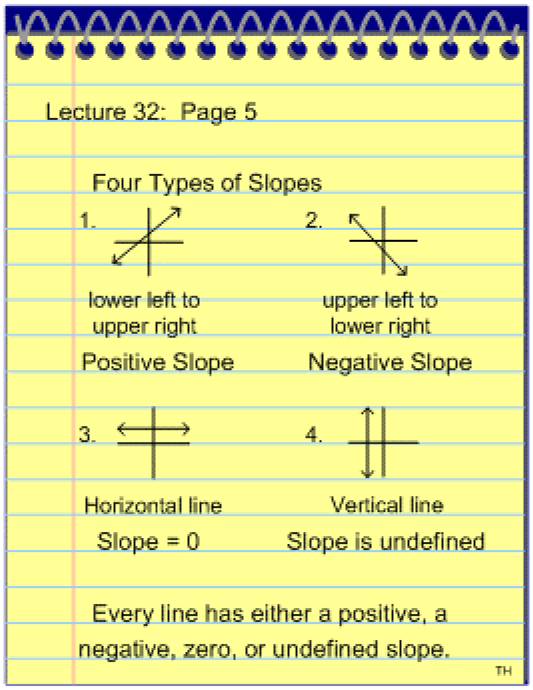
$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{2 - (-3)}{5 - 3} = \frac{5}{2}$$

You must subtract the x and y coordinates in the same order to get the correct slope.

Every time you go over two units, you go up five units.

Lecture 32 Notes, Continued

CEMP032-05



Lecture 32: Page 5

Four Types of Slopes

lower left to
upper right

upper left to
lower right

Positive Slope

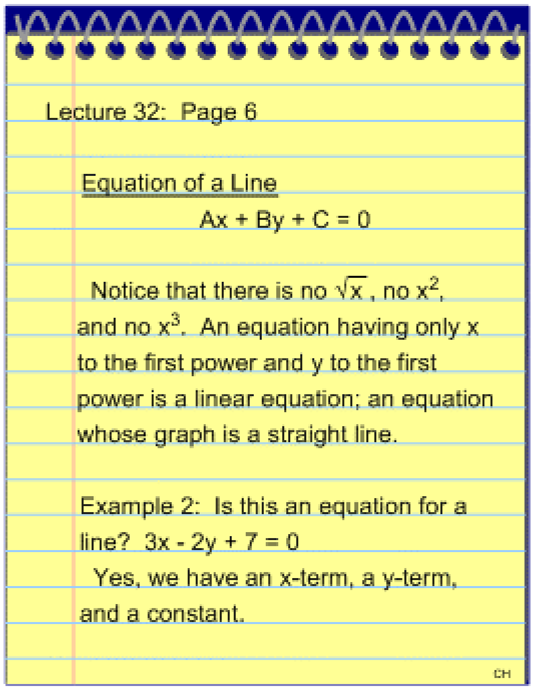
Negative Slope

Horizontal line
Slope = 0

Vertical line
Slope is undefined

Every line has either a positive, a negative, zero, or undefined slope.

CEMP032-06



Lecture 32: Page 6

Equation of a Line

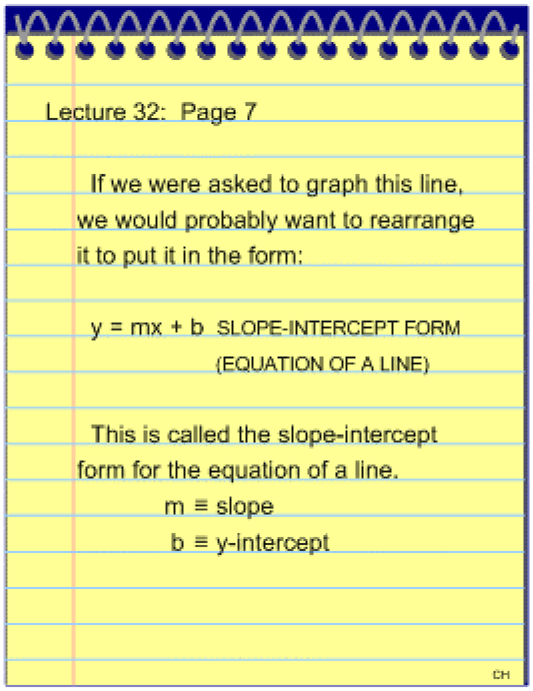
$$Ax + By + C = 0$$

Notice that there is no $\sqrt{x}$, no x^2, and no x^3. An equation having only x to the first power and y to the first power is a linear equation; an equation whose graph is a straight line.

Example 2: Is this an equation for a line? $3x - 2y + 7 = 0$

Yes, we have an x-term, a y-term, and a constant.

CEMP032-07



Lecture 32: Page 7

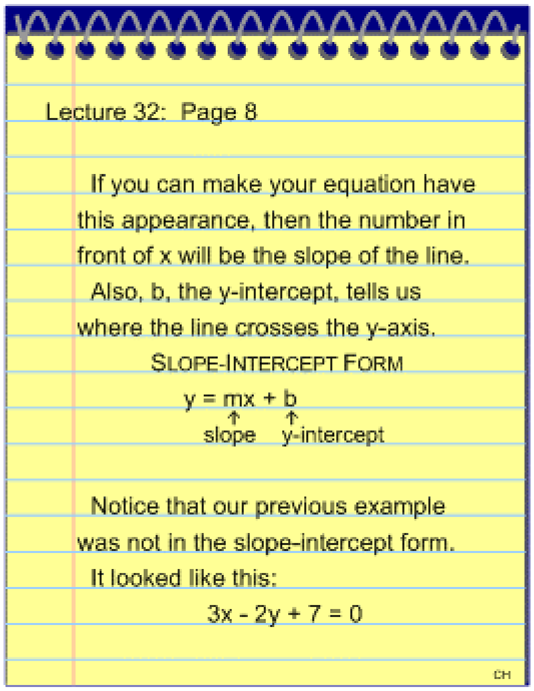
If we were asked to graph this line, we would probably want to rearrange it to put it in the form:

$y = mx + b$ SLOPE-INTERCEPT FORM
(EQUATION OF A LINE)

This is called the slope-intercept form for the equation of a line.

m ≡ slope
b ≡ y-intercept

CEMP032-08



Lecture 32: Page 8

If you can make your equation have this appearance, then the number in front of x will be the slope of the line.

Also, b, the y-intercept, tells us where the line crosses the y-axis.

SLOPE-INTERCEPT FORM

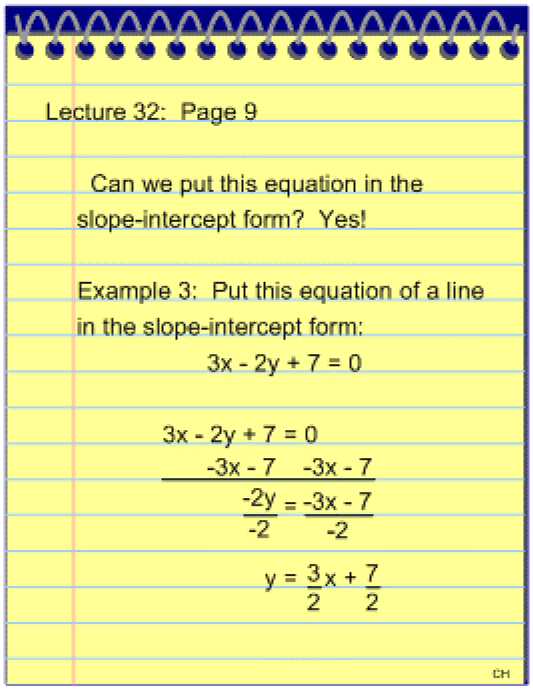
$$y = \underset{\underset{\text{slope}}{\uparrow}}{m}x + \underset{\underset{\text{y-intercept}}{\uparrow}}{b}$$

Notice that our previous example was not in the slope-intercept form. It looked like this:

$$3x - 2y + 7 = 0$$

Lecture 32 Notes, Continued

CEMP032-09



Lecture 32: Page 9

Can we put this equation in the slope-intercept form? Yes!

Example 3: Put this equation of a line in the slope-intercept form:

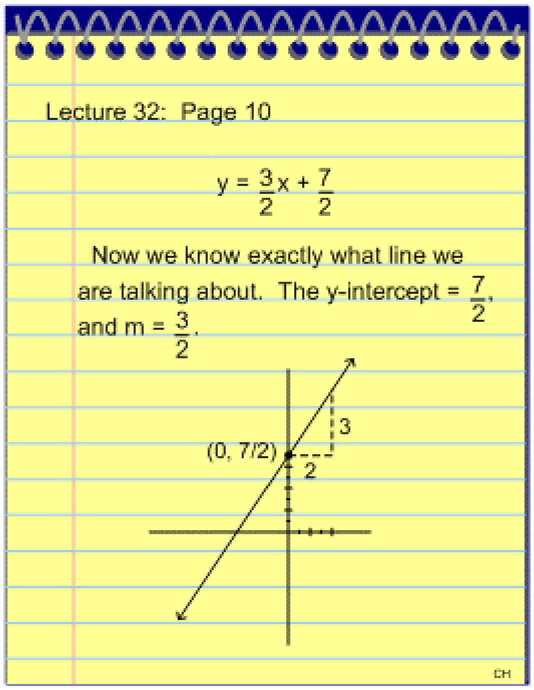
$$3x - 2y + 7 = 0$$

$$\begin{array}{r} 3x - 2y + 7 = 0 \\ \underline{-3x - 7 \quad -3x - 7} \\ \frac{-2y}{-2} = \frac{-3x - 7}{-2} \end{array}$$

$$y = \frac{3}{2}x + \frac{7}{2}$$

CH

CEMP032-10



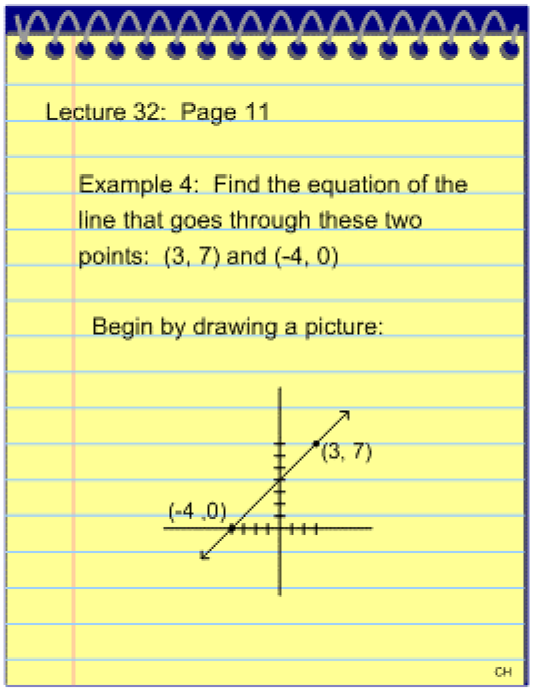
Lecture 32: Page 10

$$y = \frac{3}{2}x + \frac{7}{2}$$

Now we know exactly what line we are talking about. The y-intercept = $\frac{7}{2}$, and m = $\frac{3}{2}$.

CH

CEMP032-11



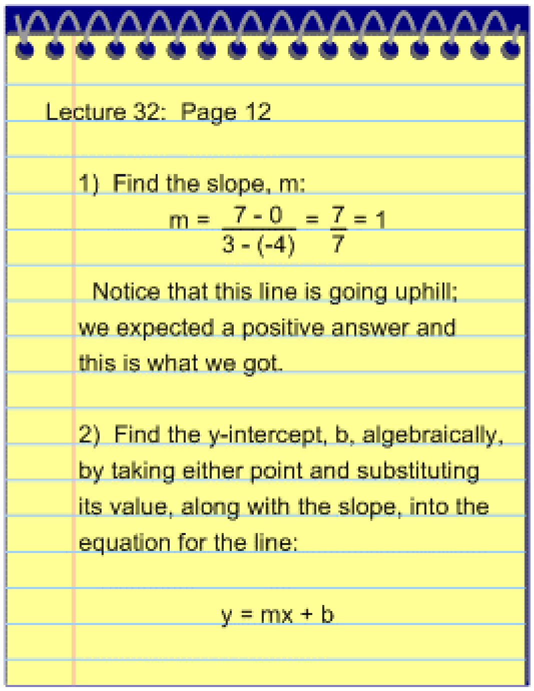
Lecture 32: Page 11

Example 4: Find the equation of the line that goes through these two points: (3, 7) and (-4, 0)

Begin by drawing a picture:

CH

CEMP032-12



Lecture 32: Page 12

1) Find the slope, m:

$$m = \frac{7 - 0}{3 - (-4)} = \frac{7}{7} = 1$$

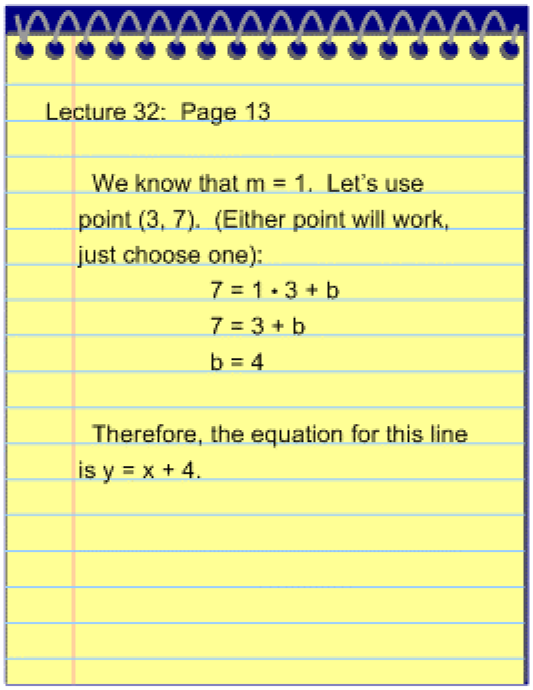
Notice that this line is going uphill; we expected a positive answer and this is what we got.

2) Find the y-intercept, b, algebraically, by taking either point and substituting its value, along with the slope, into the equation for the line:

$$y = mx + b$$

Lecture 32 Notes, Continued

CEMP032-13



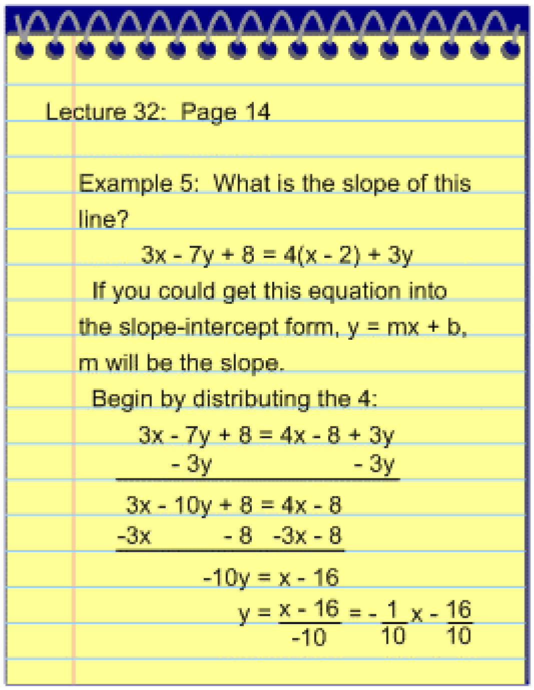
Lecture 32: Page 13

We know that $m = 1$. Let's use point (3, 7). (Either point will work, just choose one):

$$7 = 1 \cdot 3 + b$$
$$7 = 3 + b$$
$$b = 4$$

Therefore, the equation for this line is $y = x + 4$.

CEMP032-14



Lecture 32: Page 14

Example 5: What is the slope of this line?

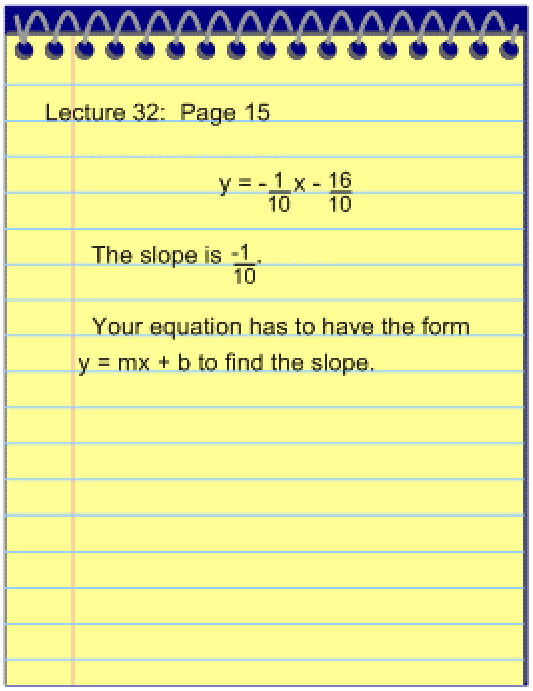
$$3x - 7y + 8 = 4(x - 2) + 3y$$

If you could get this equation into the slope-intercept form, $y = mx + b$, m will be the slope.

Begin by distributing the 4:

$$3x - 7y + 8 = 4x - 8 + 3y$$
$$\underline{\quad - 3y \qquad\qquad\qquad - 3y}$$
$$3x - 10y + 8 = 4x - 8$$
$$\underline{-3x \qquad - 8 \quad -3x - 8}$$
$$-10y = x - 16$$
$$y = \frac{x - 16}{-10} = -\frac{1}{10}x - \frac{16}{10}$$

CEMP032-15



Lecture 32: Page 15

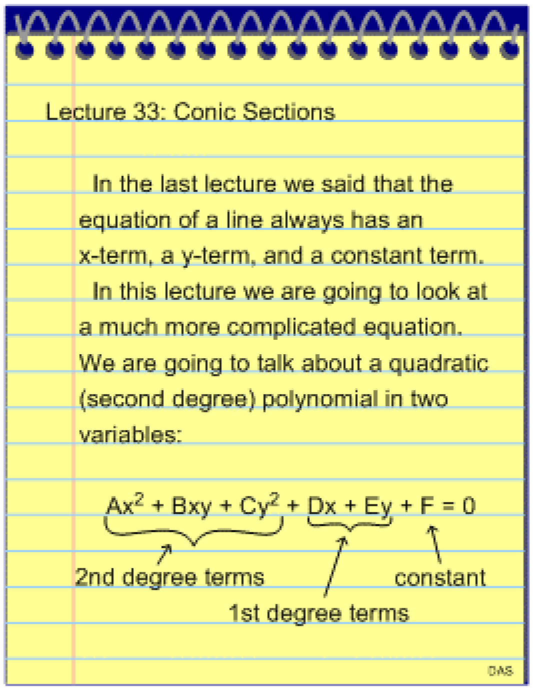
$$y = -\frac{1}{10}x - \frac{16}{10}$$

The slope is $\frac{-1}{10}$.

Your equation has to have the form $y = mx + b$ to find the slope.

Lecture 33 Notes

CEMP033-01



Lecture 33: Conic Sections

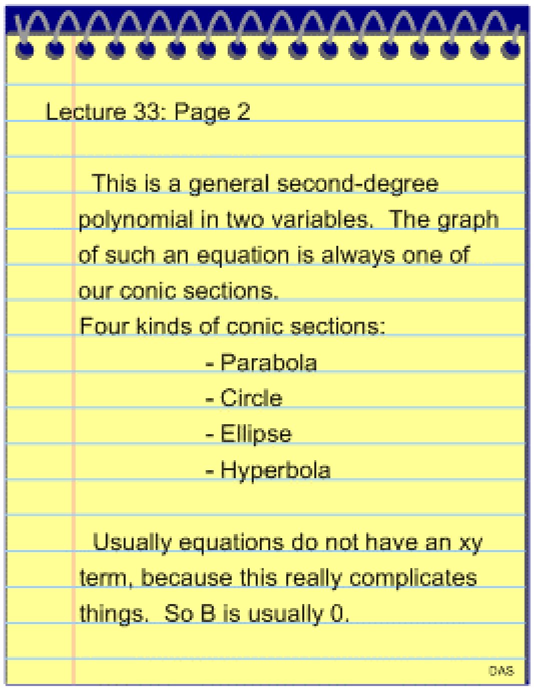
In the last lecture we said that the equation of a line always has an x-term, a y-term, and a constant term.

In this lecture we are going to look at a much more complicated equation. We are going to talk about a quadratic (second degree) polynomial in two variables:

$$\underbrace{Ax^2 + Bxy + Cy^2}_{\text{2nd degree terms}} + \underbrace{Dx + Ey}_{\text{1st degree terms}} + \underset{\text{constant}}{F} = 0$$

DAS

CEMP033-02



Lecture 33: Page 2

This is a general second-degree polynomial in two variables. The graph of such an equation is always one of our conic sections.

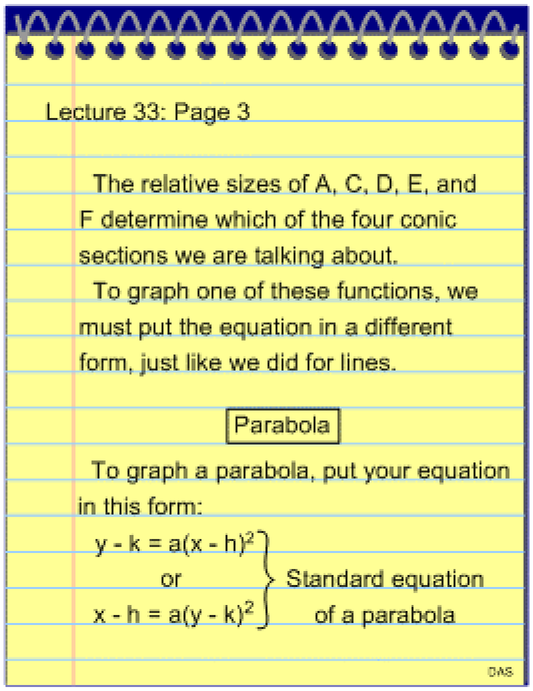
Four kinds of conic sections:

- Parabola
- Circle
- Ellipse
- Hyperbola

Usually equations do not have an xy term, because this really complicates things. So B is usually 0.

DAS

CEMP033-03



Lecture 33: Page 3

The relative sizes of A, C, D, E, and F determine which of the four conic sections we are talking about.

To graph one of these functions, we must put the equation in a different form, just like we did for lines.

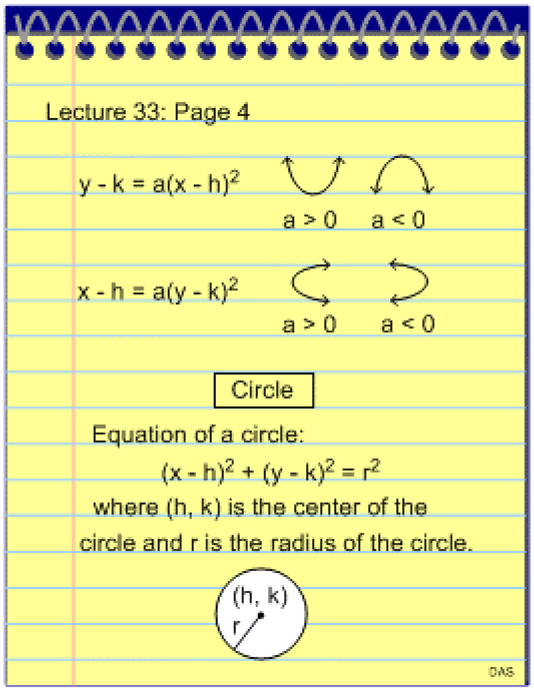
Parabola

To graph a parabola, put your equation in this form:

$$\left.\begin{array}{c} y - k = a(x - h)^2 \\ \text{or} \\ x - h = a(y - k)^2 \end{array}\right\} \text{Standard equation of a parabola}$$

DAS

CEMP033-04



Lecture 33: Page 4

$y - k = a(x - h)^2$

$a > 0$ $a < 0$

$x - h = a(y - k)^2$

$a > 0$ $a < 0$

Circle

Equation of a circle:

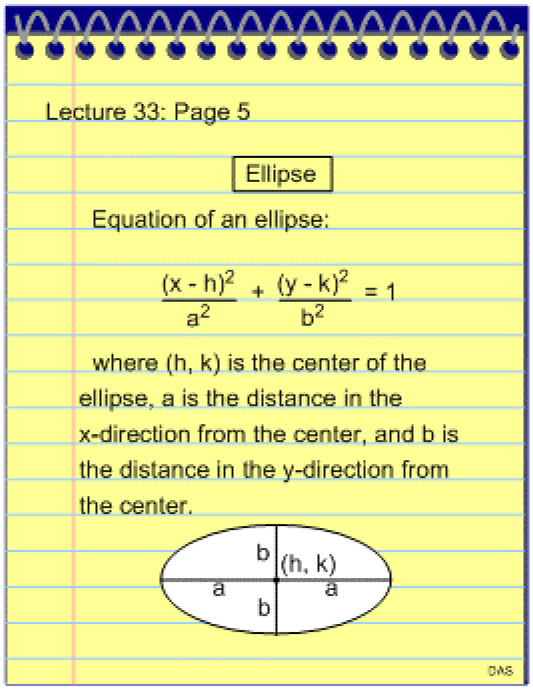
$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the center of the circle and r is the radius of the circle.

DAS

Lecture 33 Notes, Continued

CEMP033-05



Lecture 33: Page 5

Ellipse

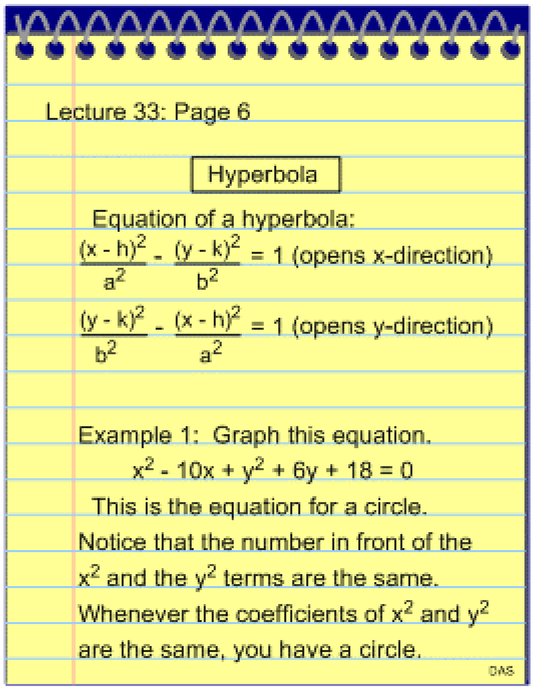
Equation of an ellipse:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

where (h, k) is the center of the ellipse, a is the distance in the x-direction from the center, and b is the distance in the y-direction from the center.

DAS

CEMP033-06



Lecture 33: Page 6

Hyperbola

Equation of a hyperbola:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \text{ (opens x-direction)}$$

$$\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1 \text{ (opens y-direction)}$$

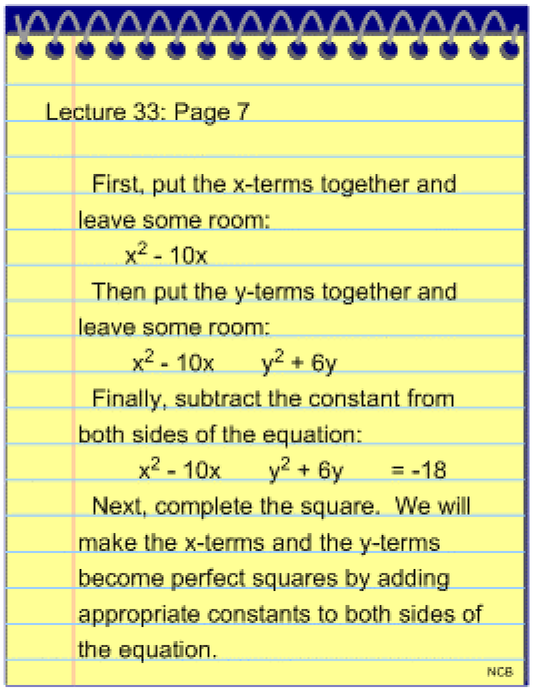
Example 1: Graph this equation.

$$x^2 - 10x + y^2 + 6y + 18 = 0$$

This is the equation for a circle. Notice that the number in front of the x^2 and the y^2 terms are the same. Whenever the coefficients of x^2 and y^2 are the same, you have a circle.

DAS

CEMP033-07



Lecture 33: Page 7

First, put the x-terms together and leave some room:

$x^2 - 10x$

Then put the y-terms together and leave some room:

$x^2 - 10x \qquad y^2 + 6y$

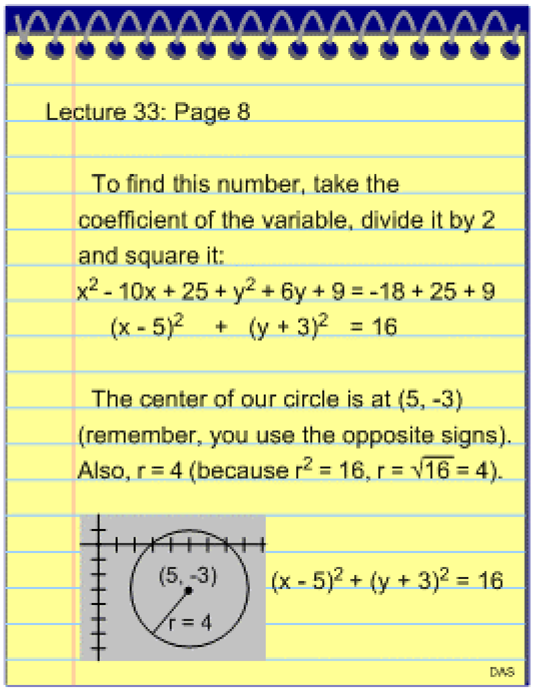
Finally, subtract the constant from both sides of the equation:

$x^2 - 10x \qquad y^2 + 6y \qquad = -18$

Next, complete the square. We will make the x-terms and the y-terms become perfect squares by adding appropriate constants to both sides of the equation.

NCB

CEMP033-08



Lecture 33: Page 8

To find this number, take the coefficient of the variable, divide it by 2 and square it:

$x^2 - 10x + 25 + y^2 + 6y + 9 = -18 + 25 + 9$

$(x - 5)^2 \quad + \quad (y + 3)^2 \quad = 16$

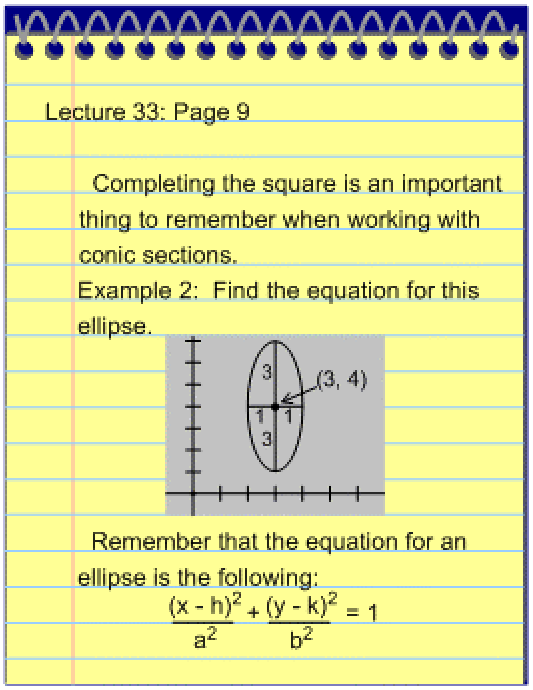
The center of our circle is at (5, -3) (remember, you use the opposite signs). Also, r = 4 (because $r^2 = 16$, $r = \sqrt{16} = 4$).

$(x - 5)^2 + (y + 3)^2 = 16$

DAS

Lecture 33 Notes, Continued

CEMP033-09



Lecture 33: Page 9

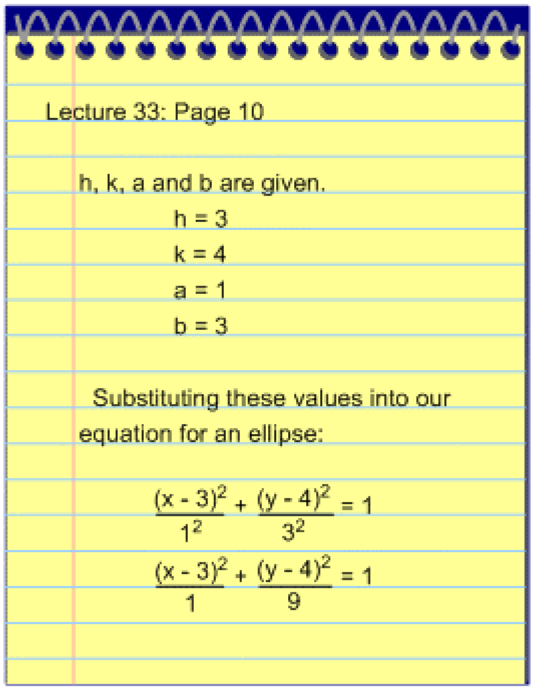
Completing the square is an important thing to remember when working with conic sections.

Example 2: Find the equation for this ellipse.

Remember that the equation for an ellipse is the following:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

CEMP033-10



Lecture 33: Page 10

h, k, a and b are given.

h = 3

k = 4

a = 1

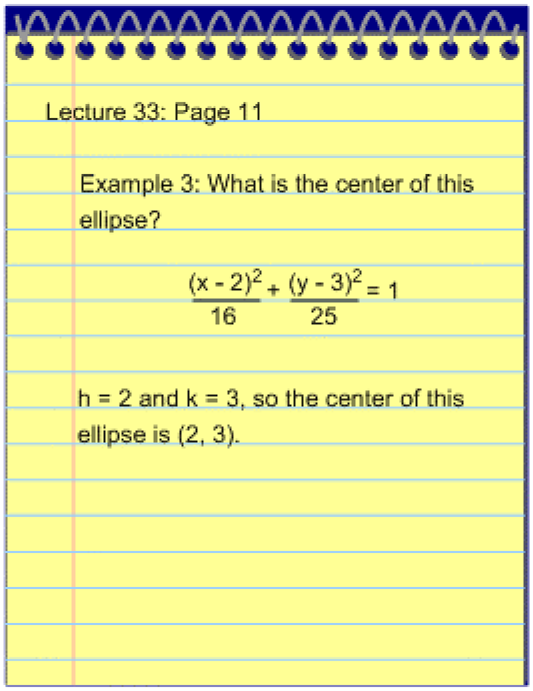
b = 3

Substituting these values into our equation for an ellipse:

$$\frac{(x-3)^2}{1^2} + \frac{(y-4)^2}{3^2} = 1$$

$$\frac{(x-3)^2}{1} + \frac{(y-4)^2}{9} = 1$$

CEMP033-11



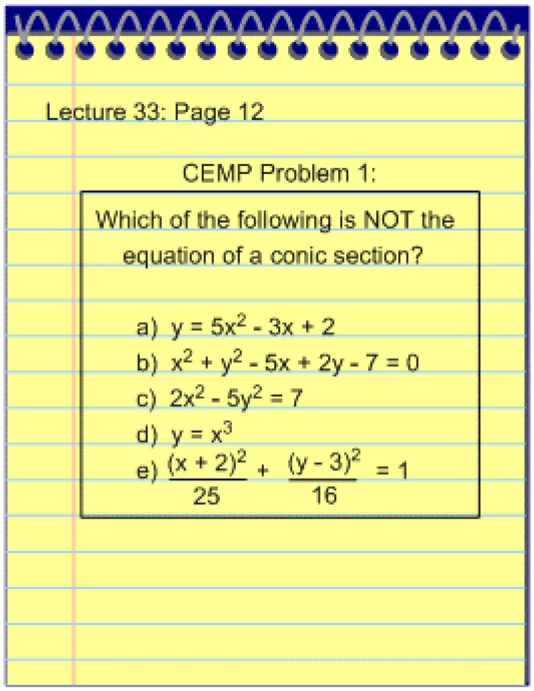
Lecture 33: Page 11

Example 3: What is the center of this ellipse?

$$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{25} = 1$$

h = 2 and k = 3, so the center of this ellipse is (2, 3).

CEMP033-12



Lecture 33: Page 12

CEMP Problem 1:

Which of the following is NOT the equation of a conic section?

a) $y = 5x^2 - 3x + 2$

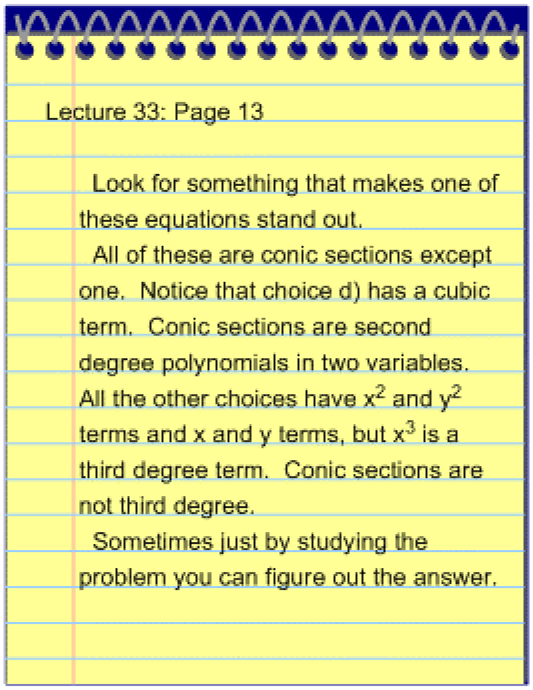
b) $x^2 + y^2 - 5x + 2y - 7 = 0$

c) $2x^2 - 5y^2 = 7$

d) $y = x^3$

e) $\frac{(x+2)^2}{25} + \frac{(y-3)^2}{16} = 1$

CEMP033-13



Lecture 33: Page 13

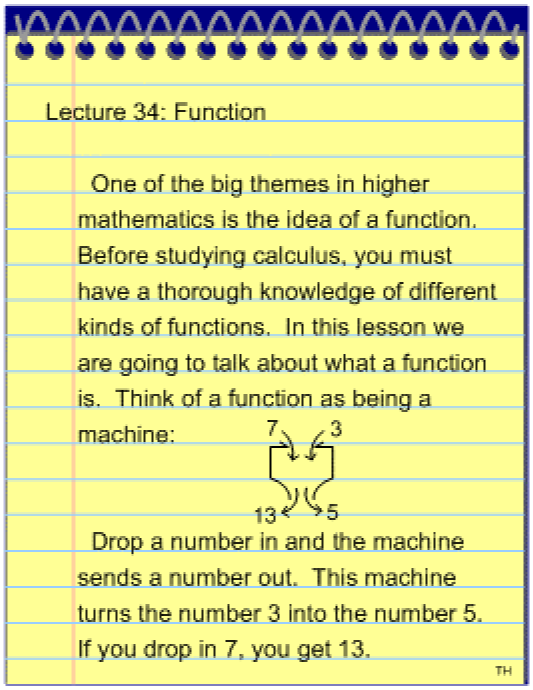
Look for something that makes one of these equations stand out.

All of these are conic sections except one. Notice that choice d) has a cubic term. Conic sections are second degree polynomials in two variables. All the other choices have x^2 and y^2 terms and x and y terms, but x^3 is a third degree term. Conic sections are not third degree.

Sometimes just by studying the problem you can figure out the answer.

Lecture 34 Notes

CEMP034-01

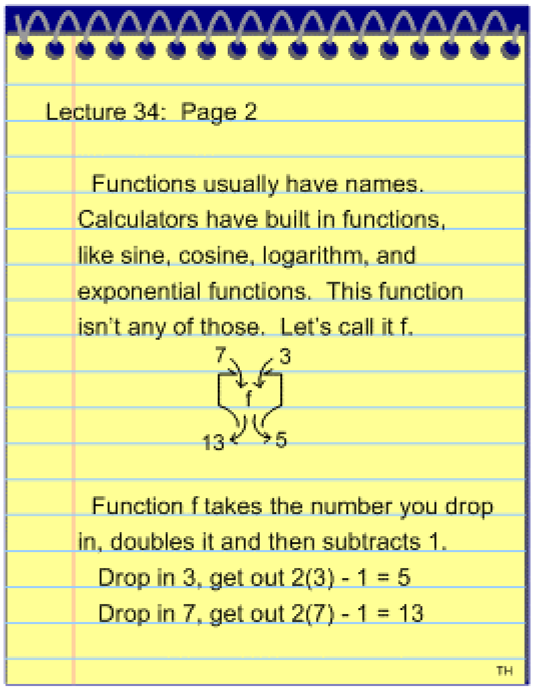


Lecture 34: Function

One of the big themes in higher mathematics is the idea of a function. Before studying calculus, you must have a thorough knowledge of different kinds of functions. In this lesson we are going to talk about what a function is. Think of a function as being a machine:

Drop a number in and the machine sends a number out. This machine turns the number 3 into the number 5. If you drop in 7, you get 13.

CEMP034-02



Lecture 34: Page 2

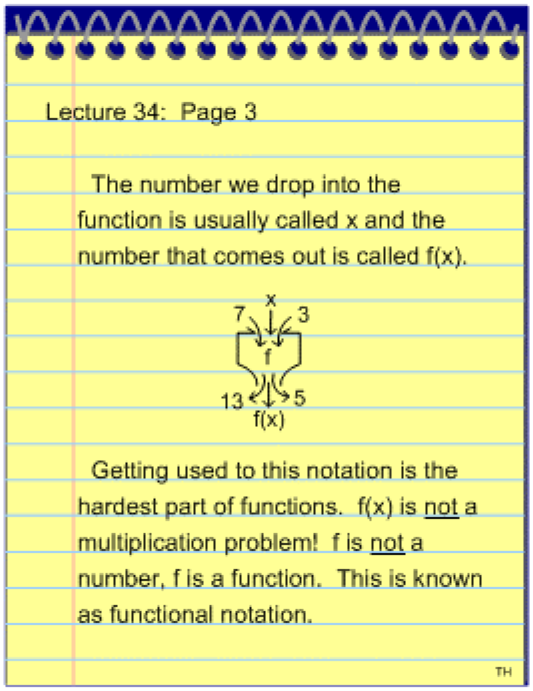
Functions usually have names. Calculators have built in functions, like sine, cosine, logarithm, and exponential functions. This function isn't any of those. Let's call it f.

Function f takes the number you drop in, doubles it and then subtracts 1.

Drop in 3, get out 2(3) - 1 = 5

Drop in 7, get out 2(7) - 1 = 13

CEMP034-03

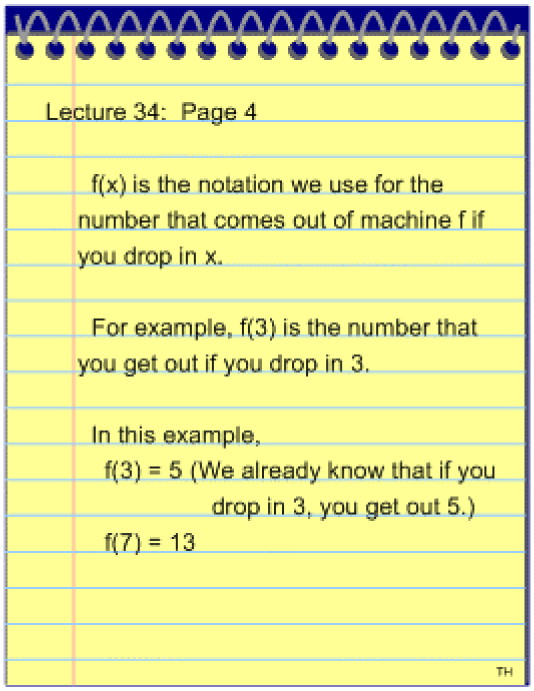


Lecture 34: Page 3

The number we drop into the function is usually called x and the number that comes out is called f(x).

Getting used to this notation is the hardest part of functions. f(x) is <u>not</u> a multiplication problem! f is <u>not</u> a number, f is a function. This is known as functional notation.

CEMP034-04



Lecture 34: Page 4

f(x) is the notation we use for the number that comes out of machine f if you drop in x.

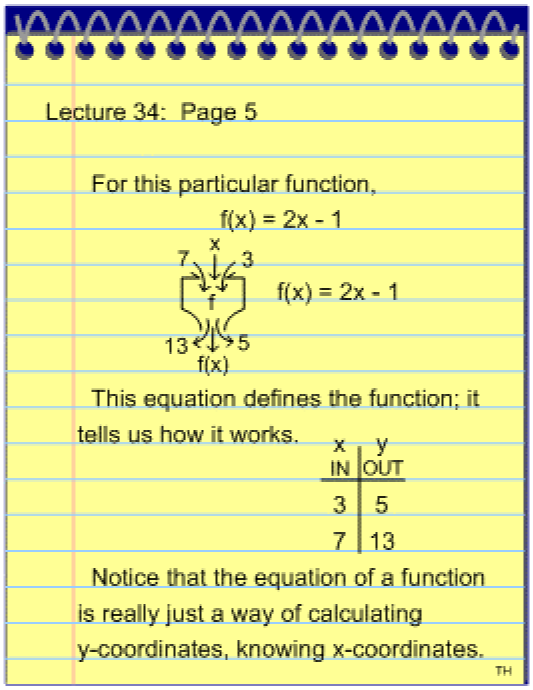
For example, f(3) is the number that you get out if you drop in 3.

In this example,

f(3) = 5 (We already know that if you drop in 3, you get out 5.)

f(7) = 13

CEMP034-05



Lecture 34: Page 5

For this particular function,

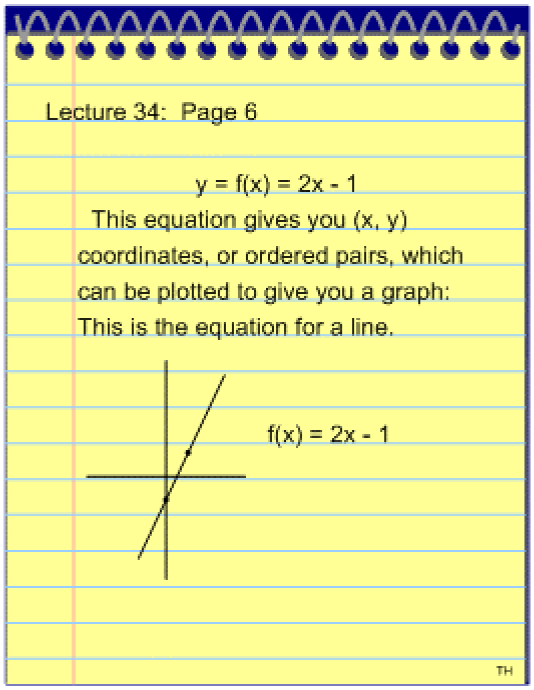
$f(x) = 2x - 1$

This equation defines the function; it tells us how it works.

x IN	y OUT
3	5
7	13

Notice that the equation of a function is really just a way of calculating y-coordinates, knowing x-coordinates.

CEMP034-06

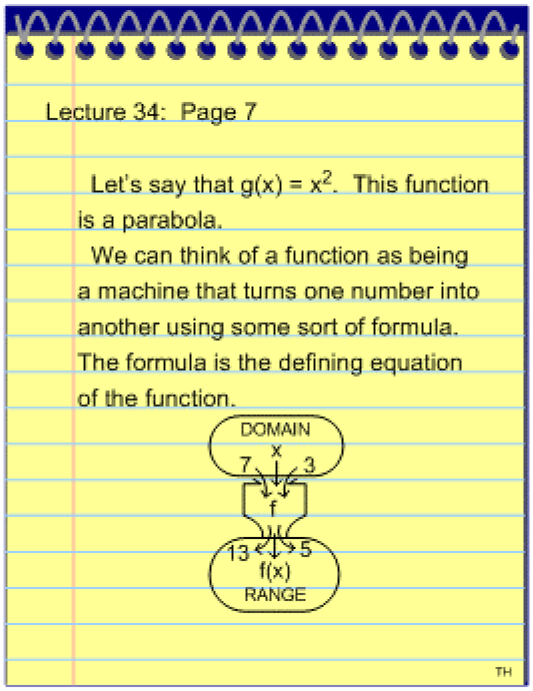


Lecture 34: Page 6

$y = f(x) = 2x - 1$

This equation gives you (x, y) coordinates, or ordered pairs, which can be plotted to give you a graph: This is the equation for a line.

CEMP034-07

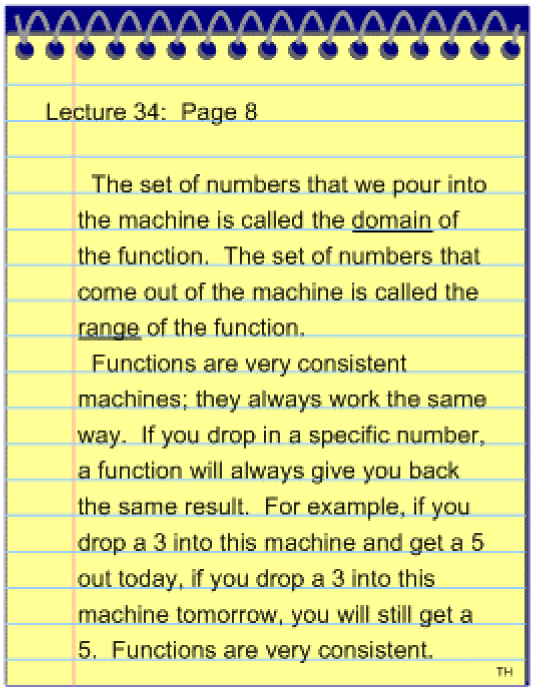


Lecture 34: Page 7

Let's say that $g(x) = x^2$. This function is a parabola.

We can think of a function as being a machine that turns one number into another using some sort of formula. The formula is the defining equation of the function.

CEMP034-08



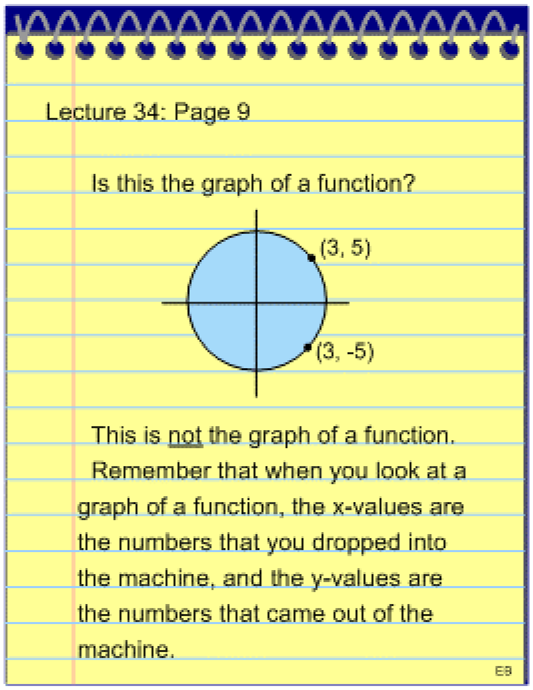
Lecture 34: Page 8

The set of numbers that we pour into the machine is called the domain of the function. The set of numbers that come out of the machine is called the range of the function.

Functions are very consistent machines; they always work the same way. If you drop in a specific number, a function will always give you back the same result. For example, if you drop a 3 into this machine and get a 5 out today, if you drop a 3 into this machine tomorrow, you will still get a 5. Functions are very consistent.

Lecture 34 Notes, Continued

CEMP034-09



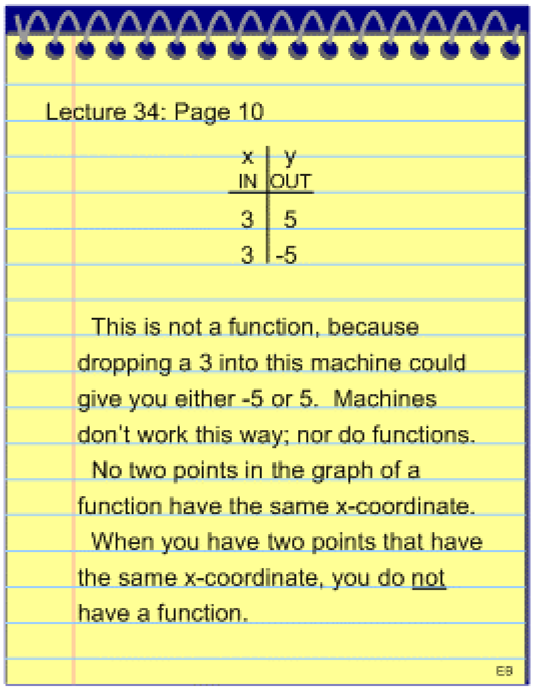
Lecture 34: Page 9

Is this the graph of a function?

This is not the graph of a function.

Remember that when you look at a graph of a function, the x-values are the numbers that you dropped into the machine, and the y-values are the numbers that came out of the machine.

CEMP034-10



Lecture 34: Page 10

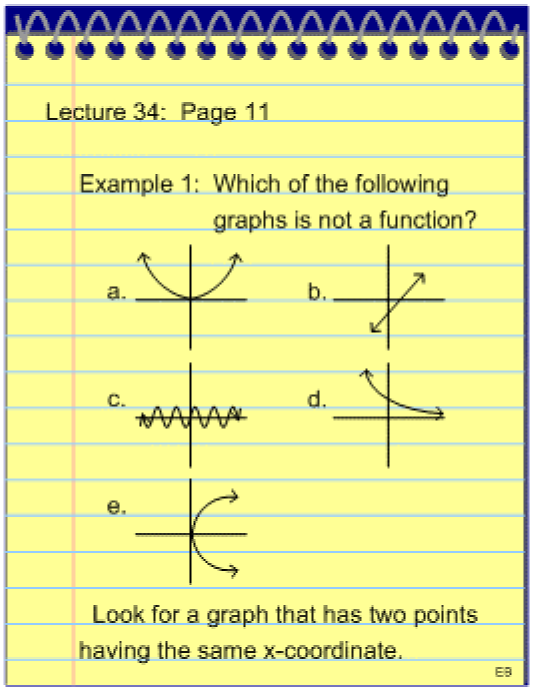
x IN	y OUT
3	5
3	-5

This is not a function, because dropping a 3 into this machine could give you either -5 or 5. Machines don't work this way; nor do functions.

No two points in the graph of a function have the same x-coordinate.

When you have two points that have the same x-coordinate, you do not have a function.

CEMP034-11

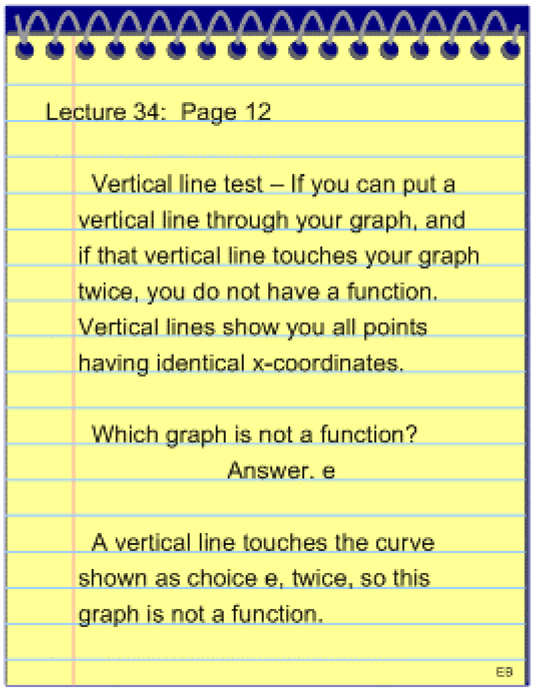


Lecture 34: Page 11

Example 1: Which of the following graphs is not a function?

Look for a graph that has two points having the same x-coordinate.

CEMP034-12



Lecture 34: Page 12

Vertical line test – If you can put a vertical line through your graph, and if that vertical line touches your graph twice, you do not have a function. Vertical lines show you all points having identical x-coordinates.

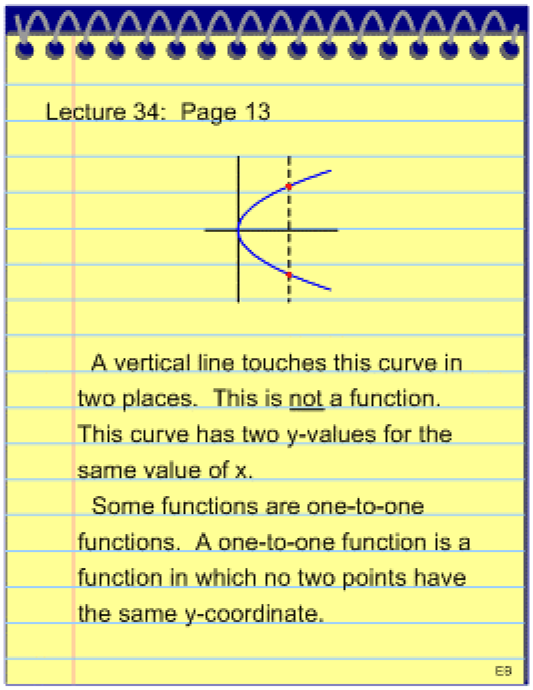
Which graph is not a function?

Answer. e

A vertical line touches the curve shown as choice e, twice, so this graph is not a function.

Lecture 34 Notes, Continued

CEMP034-13

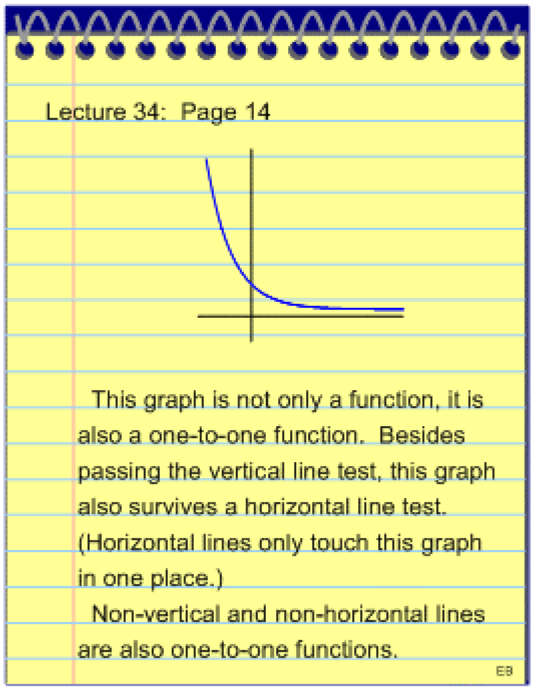


Lecture 34: Page 13

A vertical line touches this curve in two places. This is not a function. This curve has two y-values for the same value of x.

Some functions are one-to-one functions. A one-to-one function is a function in which no two points have the same y-coordinate.

CEMP034-14

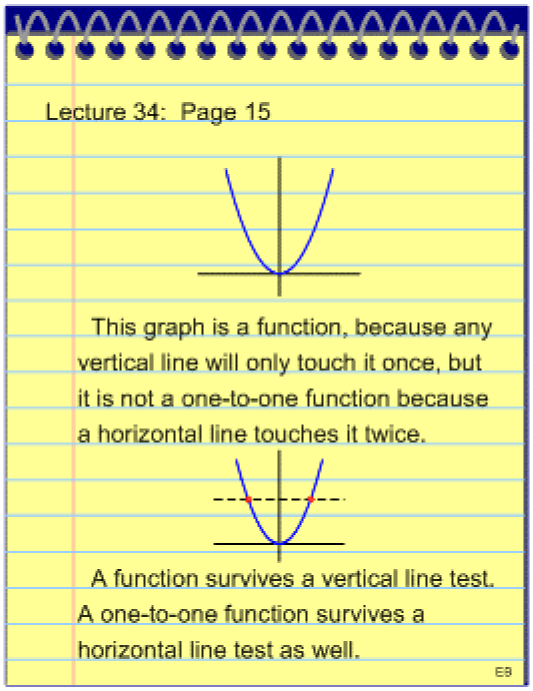


Lecture 34: Page 14

This graph is not only a function, it is also a one-to-one function. Besides passing the vertical line test, this graph also survives a horizontal line test. (Horizontal lines only touch this graph in one place.)

Non-vertical and non-horizontal lines are also one-to-one functions.

CEMP034-15

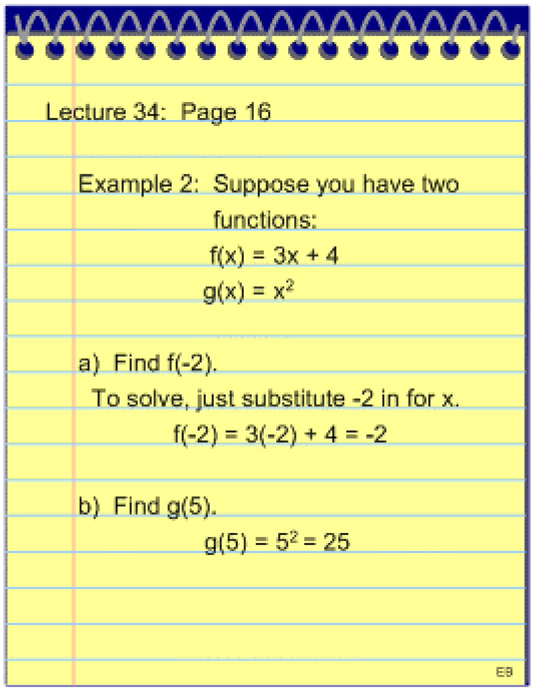


Lecture 34: Page 15

This graph is a function, because any vertical line will only touch it once, but it is not a one-to-one function because a horizontal line touches it twice.

A function survives a vertical line test. A one-to-one function survives a horizontal line test as well.

CEMP034-16



Lecture 34: Page 16

Example 2: Suppose you have two functions:

$f(x) = 3x + 4$

$g(x) = x^2$

a) Find f(-2).

To solve, just substitute -2 in for x.

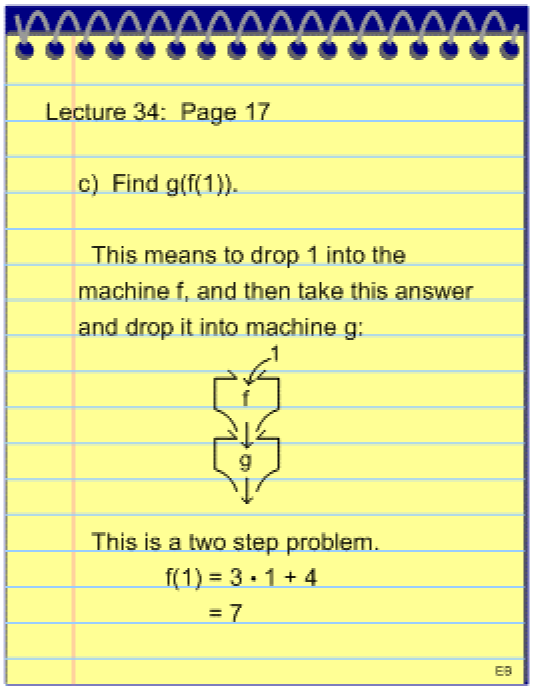
$f(-2) = 3(-2) + 4 = -2$

b) Find g(5).

$g(5) = 5^2 = 25$

Lecture 34 Notes, Continued

CEMP034-17



Lecture 34: Page 17

c) Find $g(f(1))$.

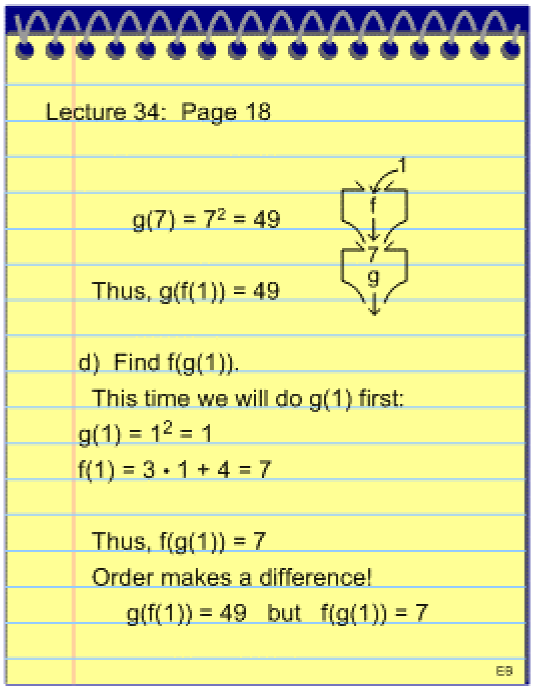
This means to drop 1 into the machine f, and then take this answer and drop it into machine g:

This is a two step problem.

$f(1) = 3 \cdot 1 + 4$

$= 7$

CEMP034-18



Lecture 34: Page 18

$g(7) = 7^2 = 49$

Thus, $g(f(1)) = 49$

d) Find $f(g(1))$.

This time we will do $g(1)$ first:

$g(1) = 1^2 = 1$

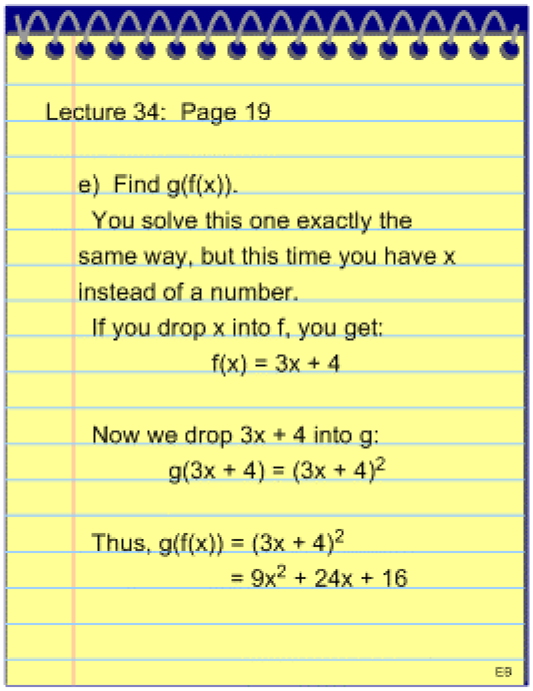
$f(1) = 3 \cdot 1 + 4 = 7$

Thus, $f(g(1)) = 7$

Order makes a difference!

$g(f(1)) = 49$ but $f(g(1)) = 7$

CEMP034-19



Lecture 34: Page 19

e) Find $g(f(x))$.

You solve this one exactly the same way, but this time you have x instead of a number.

If you drop x into f, you get:

$f(x) = 3x + 4$

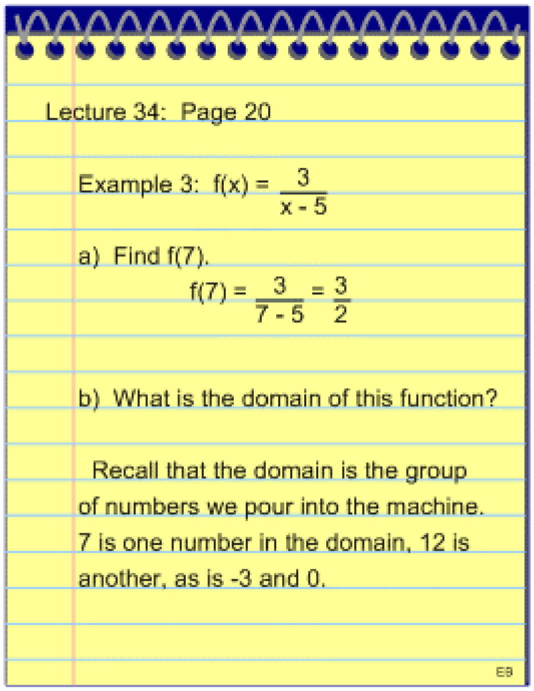
Now we drop $3x + 4$ into g:

$g(3x + 4) = (3x + 4)^2$

Thus, $g(f(x)) = (3x + 4)^2$

$= 9x^2 + 24x + 16$

CEMP034-20



Lecture 34: Page 20

Example 3: $f(x) = \frac{3}{x - 5}$

a) Find $f(7)$.

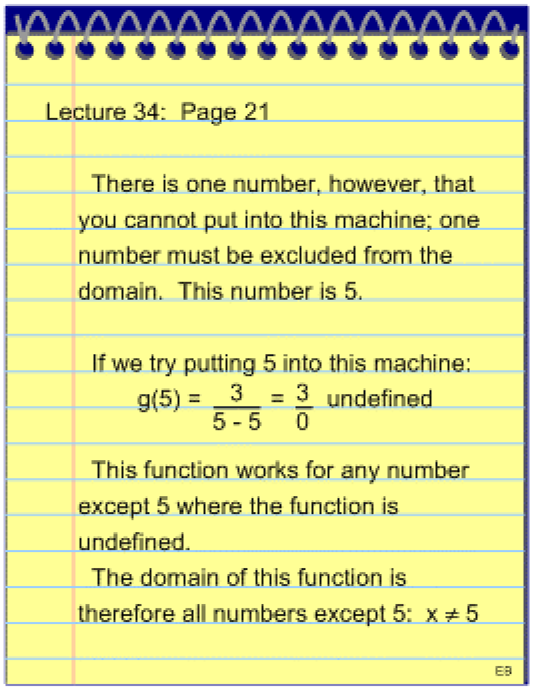
$f(7) = \frac{3}{7 - 5} = \frac{3}{2}$

b) What is the domain of this function?

Recall that the domain is the group of numbers we pour into the machine. 7 is one number in the domain, 12 is another, as is -3 and 0.

Lecture 34 Notes, Continued

CEMP034-21



Lecture 34: Page 21

There is one number, however, that you cannot put into this machine; one number must be excluded from the domain. This number is 5.

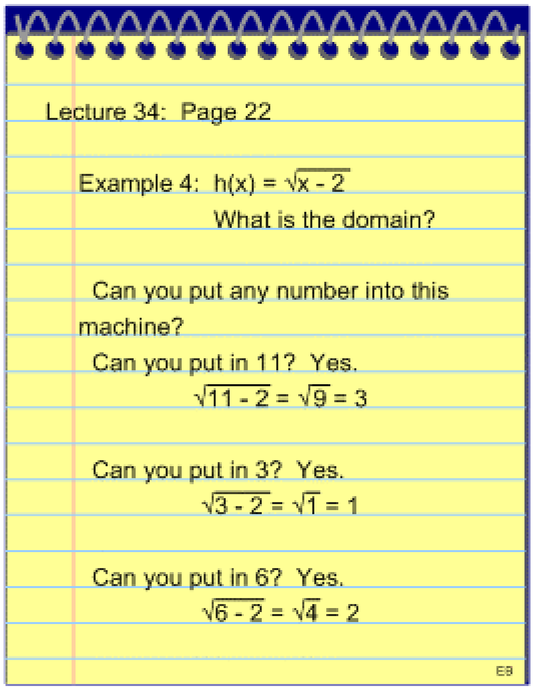
If we try putting 5 into this machine:

$$g(5) = \frac{3}{5-5} = \frac{3}{0} \text{ undefined}$$

This function works for any number except 5 where the function is undefined.

The domain of this function is therefore all numbers except 5: $x \neq 5$

CEMP034-22



Lecture 34: Page 22

Example 4: $h(x) = \sqrt{x - 2}$

What is the domain?

Can you put any number into this machine?

Can you put in 11? Yes.

$$\sqrt{11 - 2} = \sqrt{9} = 3$$

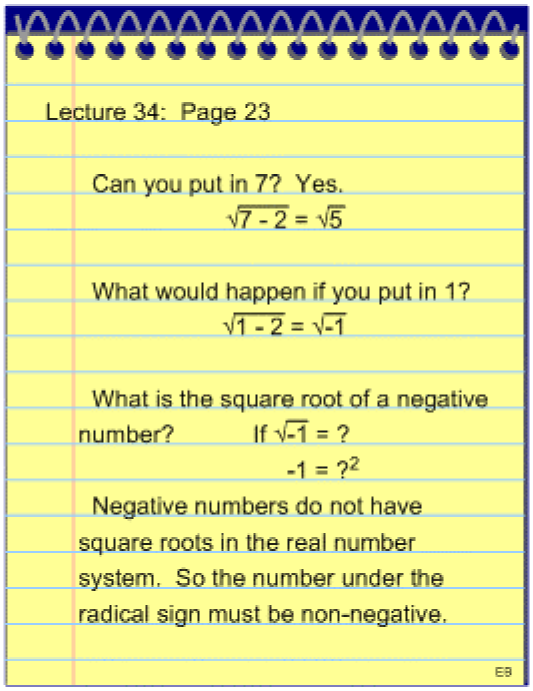
Can you put in 3? Yes.

$$\sqrt{3 - 2} = \sqrt{1} = 1$$

Can you put in 6? Yes.

$$\sqrt{6 - 2} = \sqrt{4} = 2$$

CEMP034-23



Lecture 34: Page 23

Can you put in 7? Yes.

$$\sqrt{7 - 2} = \sqrt{5}$$

What would happen if you put in 1?

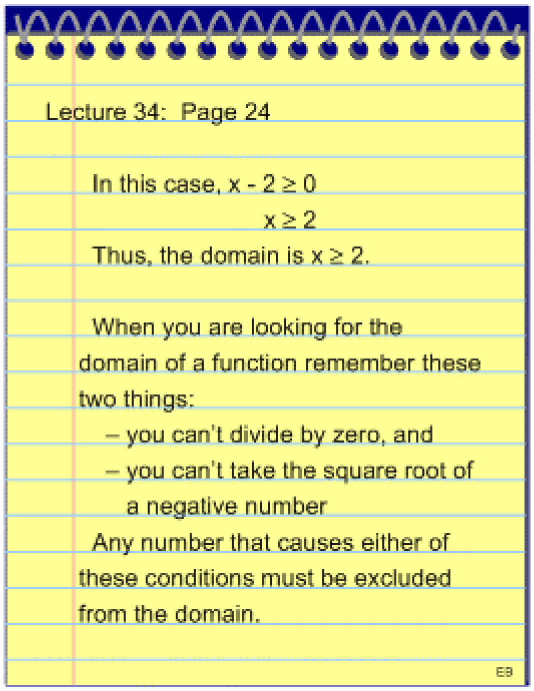
$$\sqrt{1 - 2} = \sqrt{-1}$$

What is the square root of a negative number? If $\sqrt{-1} = ?$

$$-1 = ?^2$$

Negative numbers do not have square roots in the real number system. So the number under the radical sign must be non-negative.

CEMP034-24



Lecture 34: Page 24

In this case, $x - 2 \geq 0$

$$x \geq 2$$

Thus, the domain is $x \geq 2$.

When you are looking for the domain of a function remember these two things:

- you can't divide by zero, and
- you can't take the square root of a negative number

Any number that causes either of these conditions must be excluded from the domain.

Lecture 34 Notes, Continued

CEMP034-25

Lecture 34: Page 25

CEMP Problem 1:

Which of the following does NOT define a function?

a) $y = x + 2$

b) $x = y + 2$

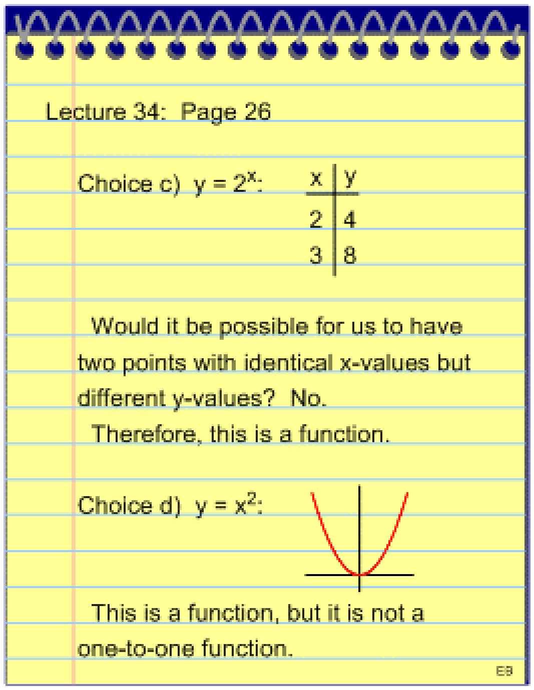
c) $y = 2^x$

d) $y = x^2$

e) $x = y^2$

Choices a and b are equations for a line. Lines are functions, they are even one-to-one functions.

CEMP034-26



Lecture 34: Page 26

Choice c) $y = 2^x$:

x	y
2	4
3	8

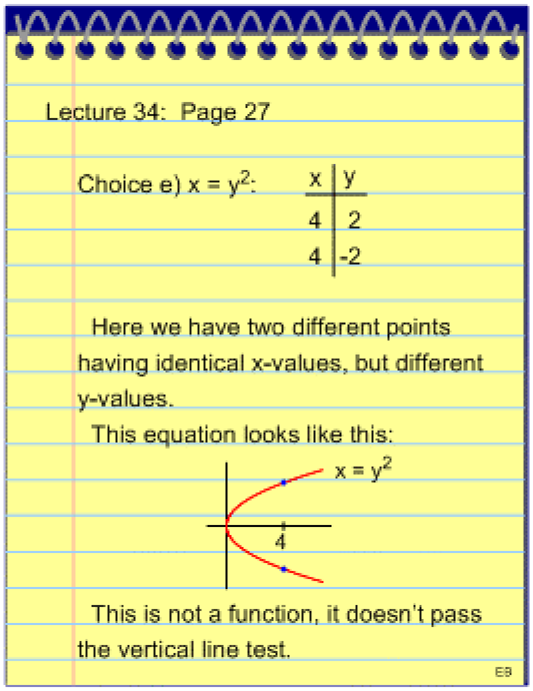
Would it be possible for us to have two points with identical x-values but different y-values? No.

Therefore, this is a function.

Choice d) $y = x^2$:

This is a function, but it is not a one-to-one function.

CEMP034-27



Lecture 34: Page 27

Choice e) $x = y^2$:

x	y
4	2
4	-2

Here we have two different points having identical x-values, but different y-values.

This equation looks like this:

This is not a function, it doesn't pass the vertical line test.

CEMP034-28

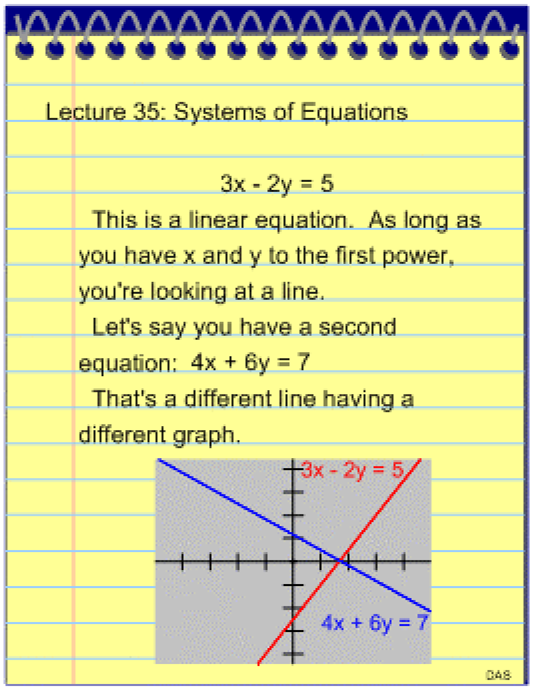
Lecture 34: Page 28

Answer. e

Any time you have two points with identical x-coordinates, but different y-coordinates you do <u>not</u> have a function.

Lecture 35 Notes

CEMP035-01



Lecture 35: Systems of Equations

$3x - 2y = 5$

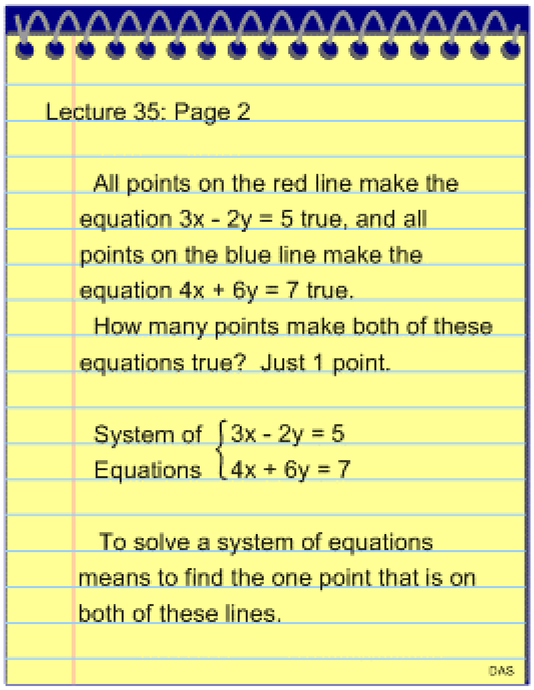
This is a linear equation. As long as you have x and y to the first power, you're looking at a line.

Let's say you have a second equation: $4x + 6y = 7$

That's a different line having a different graph.

DAS

CEMP035-02



Lecture 35: Page 2

All points on the red line make the equation $3x - 2y = 5$ true, and all points on the blue line make the equation $4x + 6y = 7$ true.

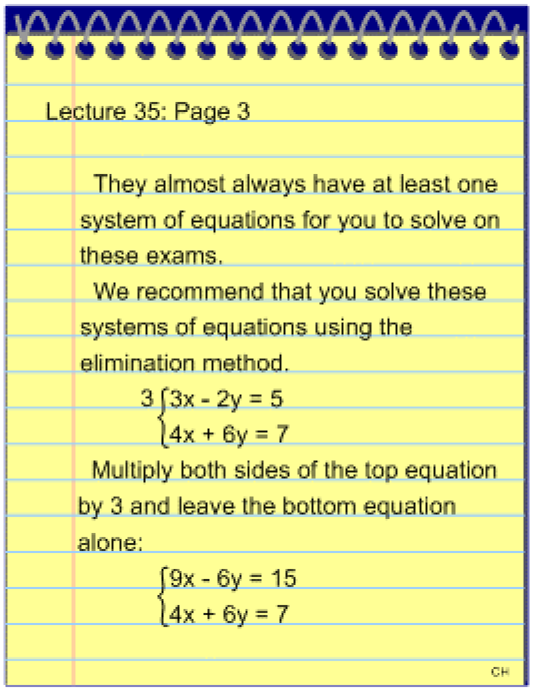
How many points make both of these equations true? Just 1 point.

System of Equations $\begin{cases} 3x - 2y = 5 \\ 4x + 6y = 7 \end{cases}$

To solve a system of equations means to find the one point that is on both of these lines.

DAS

CEMP035-03



Lecture 35: Page 3

They almost always have at least one system of equations for you to solve on these exams.

We recommend that you solve these systems of equations using the elimination method.

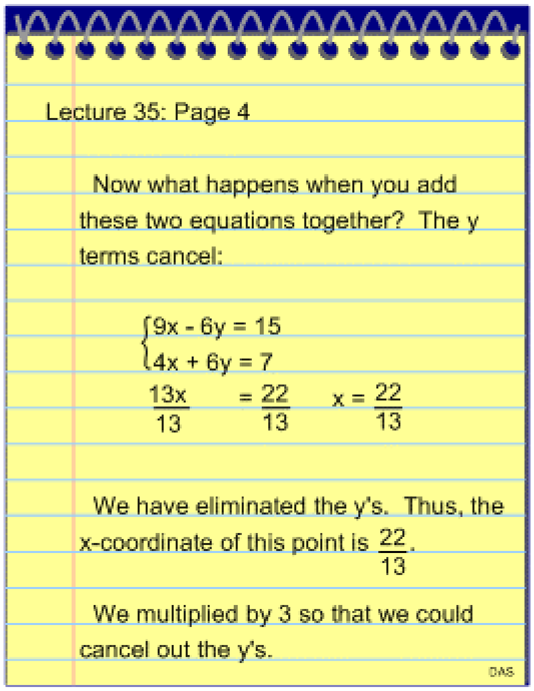
$$3\begin{cases} 3x - 2y = 5 \\ 4x + 6y = 7 \end{cases}$$

Multiply both sides of the top equation by 3 and leave the bottom equation alone:

$$\begin{cases} 9x - 6y = 15 \\ 4x + 6y = 7 \end{cases}$$

CH

CEMP035-04



Lecture 35: Page 4

Now what happens when you add these two equations together? The y terms cancel:

$$\begin{cases} 9x - 6y = 15 \\ 4x + 6y = 7 \end{cases}$$

$$\frac{13x}{13} = \frac{22}{13} \qquad x = \frac{22}{13}$$

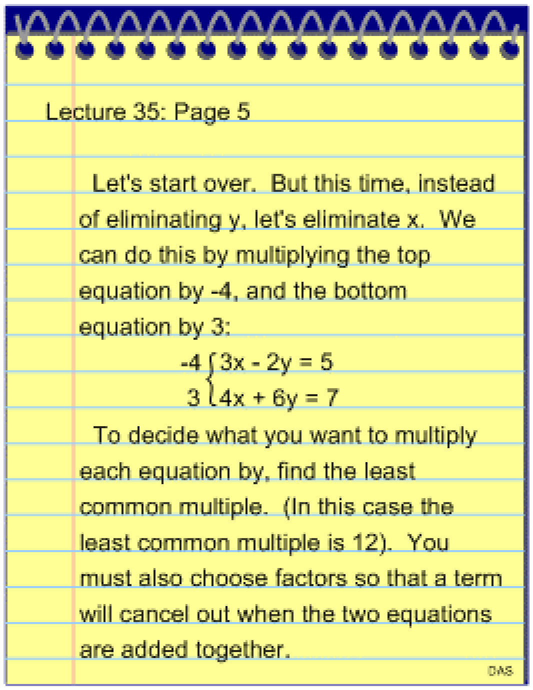
We have eliminated the y's. Thus, the x-coordinate of this point is $\frac{22}{13}$.

We multiplied by 3 so that we could cancel out the y's.

DAS

Lecture 35 Notes, Continued

CEMP035-05



Lecture 35: Page 5

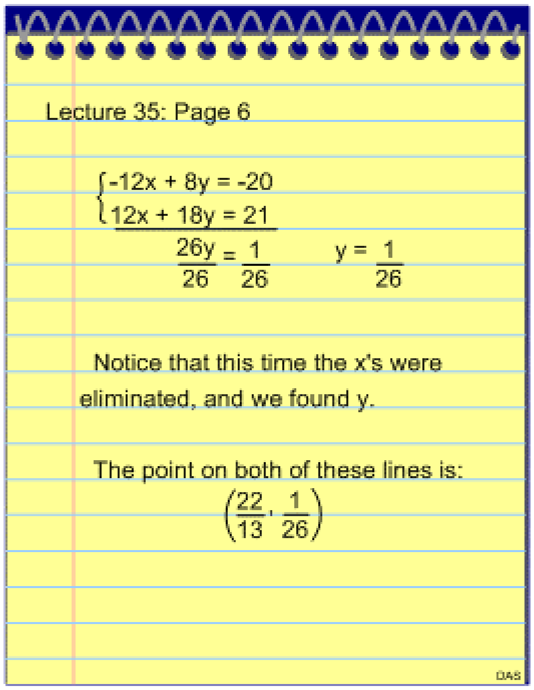
Let's start over. But this time, instead of eliminating y, let's eliminate x. We can do this by multiplying the top equation by -4, and the bottom equation by 3:

$$\begin{matrix} -4 \\ 3 \end{matrix}\begin{cases} 3x - 2y = 5 \\ 4x + 6y = 7 \end{cases}$$

To decide what you want to multiply each equation by, find the least common multiple. (In this case the least common multiple is 12). You must also choose factors so that a term will cancel out when the two equations are added together.

DAS

CEMP035-06



Lecture 35: Page 6

$$\begin{cases} -12x + 8y = -20 \\ \underline{12x + 18y = 21} \end{cases}$$

$$\frac{26y}{26} = \frac{1}{26} \qquad y = \frac{1}{26}$$

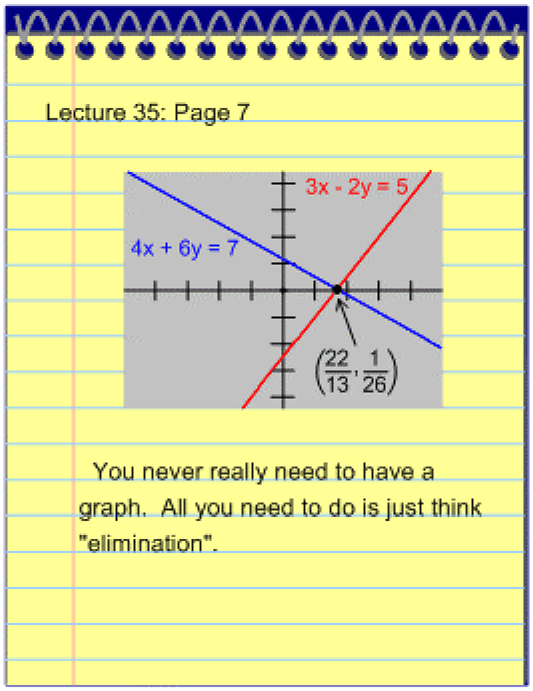
Notice that this time the x's were eliminated, and we found y.

The point on both of these lines is:

$$\left(\frac{22}{13}, \frac{1}{26}\right)$$

DAS

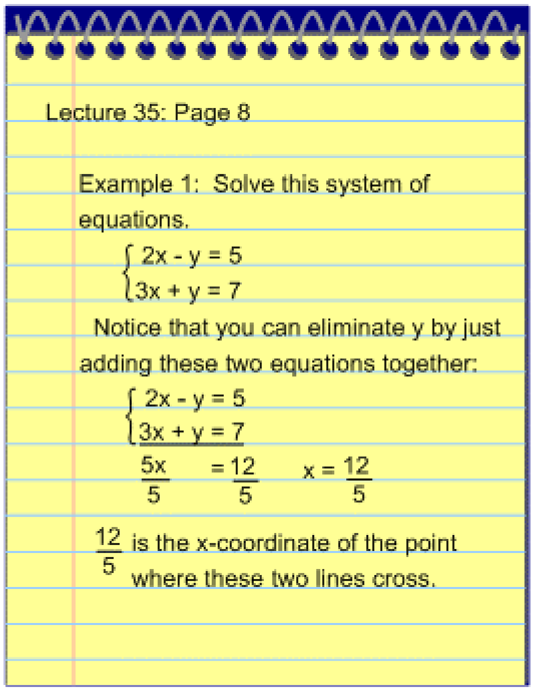
CEMP035-07



Lecture 35: Page 7

You never really need to have a graph. All you need to do is just think "elimination".

CEMP035-08



Lecture 35: Page 8

Example 1: Solve this system of equations.

$$\begin{cases} 2x - y = 5 \\ 3x + y = 7 \end{cases}$$

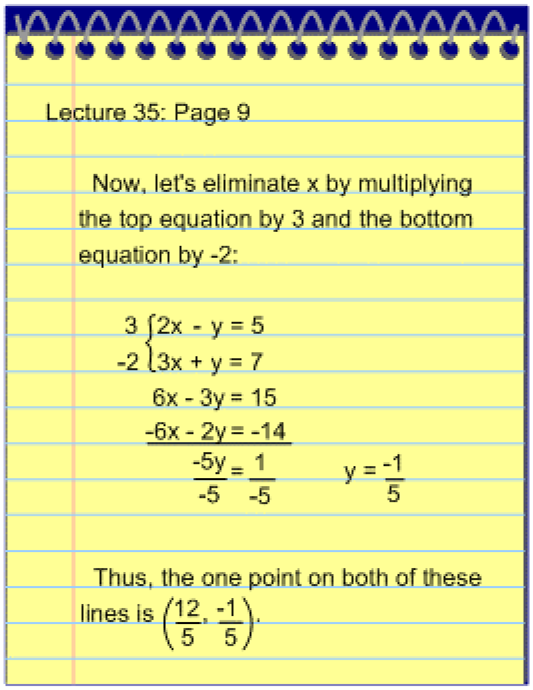
Notice that you can eliminate y by just adding these two equations together:

$$\begin{cases} 2x - y = 5 \\ \underline{3x + y = 7} \end{cases}$$

$$\frac{5x}{5} = \frac{12}{5} \qquad x = \frac{12}{5}$$

$\frac{12}{5}$ is the x-coordinate of the point where these two lines cross.

CEMP035-09



Lecture 35: Page 9

Now, let's eliminate x by multiplying the top equation by 3 and the bottom equation by -2:

$$\begin{matrix} 3 \\ -2 \end{matrix}\begin{cases} 2x - y = 5 \\ 3x + y = 7 \end{cases}$$

$$6x - 3y = 15$$

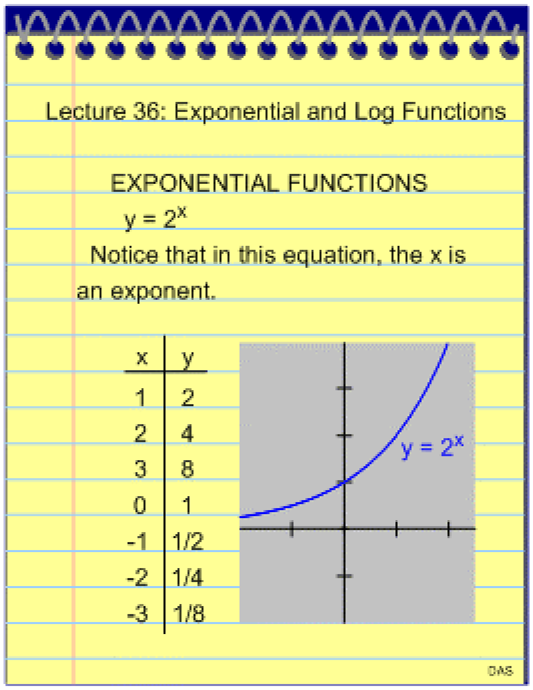
$$\underline{-6x - 2y = -14}$$

$$\frac{-5y}{-5} = \frac{1}{-5} \qquad y = \frac{-1}{5}$$

Thus, the one point on both of these lines is $\left(\frac{12}{5}, \frac{-1}{5}\right)$.

Lecture 36 Notes

CEMP036-01



Lecture 36: Exponential and Log Functions

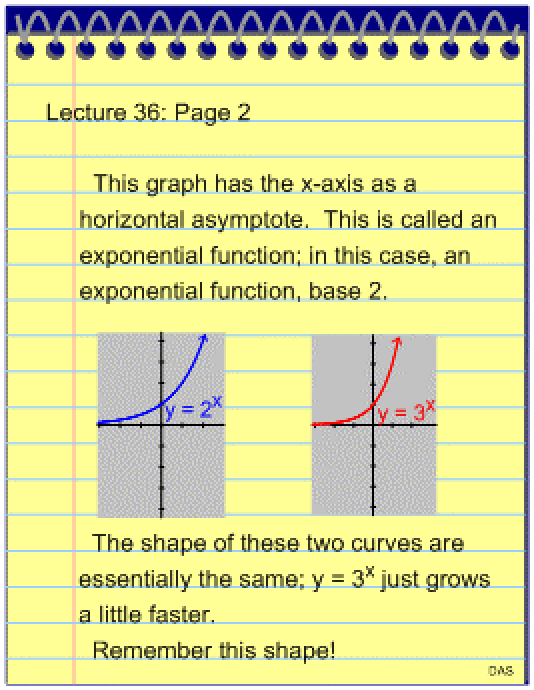
EXPONENTIAL FUNCTIONS

$y = 2^x$

Notice that in this equation, the x is an exponent.

x	y
1	2
2	4
3	8
0	1
-1	1/2
-2	1/4
-3	1/8

CEMP036-02



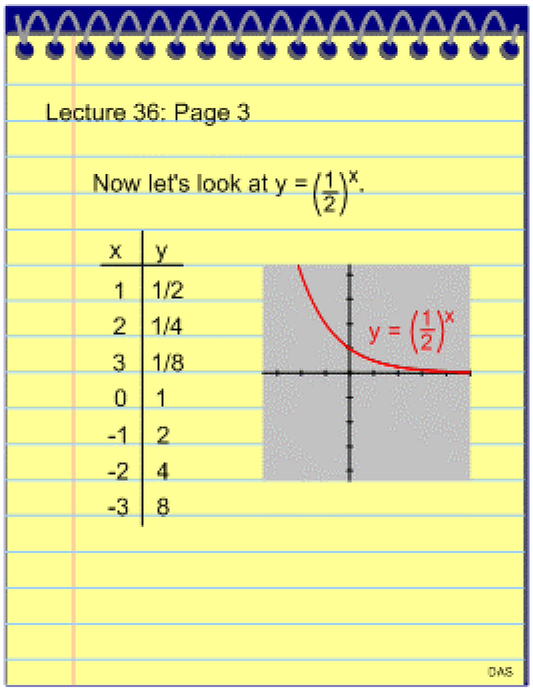
Lecture 36: Page 2

This graph has the x-axis as a horizontal asymptote. This is called an exponential function; in this case, an exponential function, base 2.

The shape of these two curves are essentially the same; $y = 3^x$ just grows a little faster.

Remember this shape!

CEMP036-03

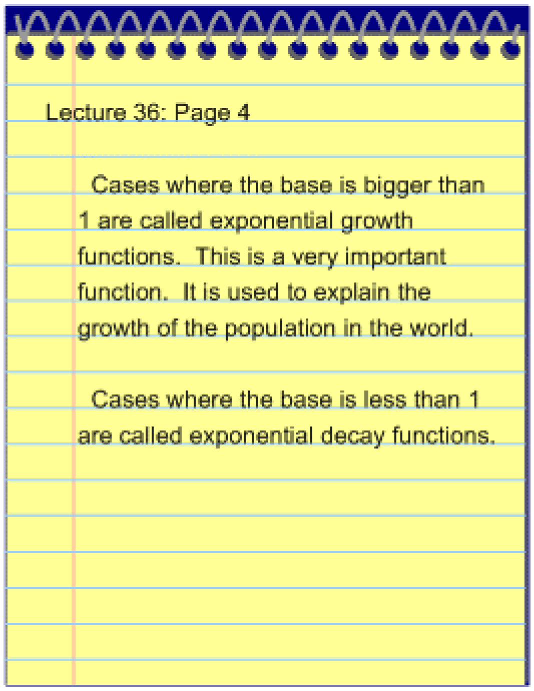


Lecture 36: Page 3

Now let's look at $y = \left(\frac{1}{2}\right)^x$.

x	y
1	1/2
2	1/4
3	1/8
0	1
-1	2
-2	4
-3	8

CEMP036-04



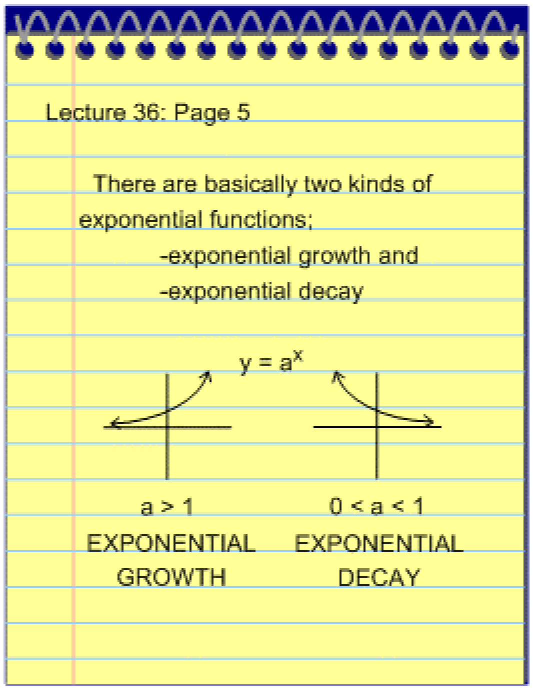
Lecture 36: Page 4

Cases where the base is bigger than 1 are called exponential growth functions. This is a very important function. It is used to explain the growth of the population in the world.

Cases where the base is less than 1 are called exponential decay functions.

Lecture 36 Notes, Continued

CEMP036-05



Lecture 36: Page 5

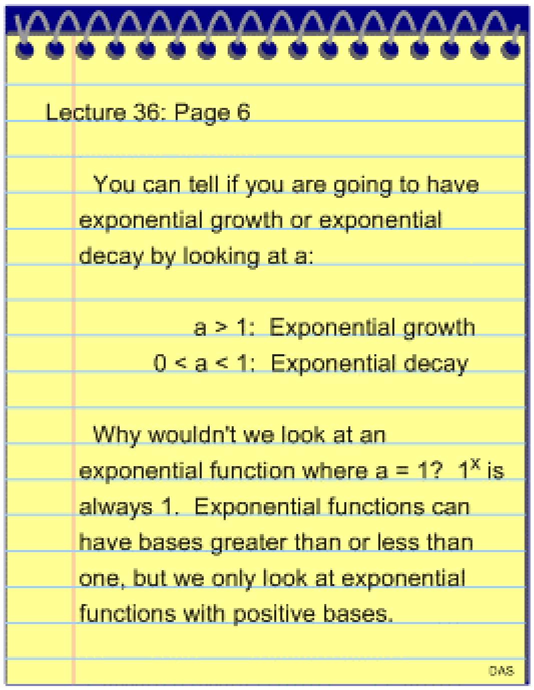
There are basically two kinds of exponential functions;

-exponential growth and

-exponential decay

$y = a^x$

CEMP036-06



Lecture 36: Page 6

You can tell if you are going to have exponential growth or exponential decay by looking at a:

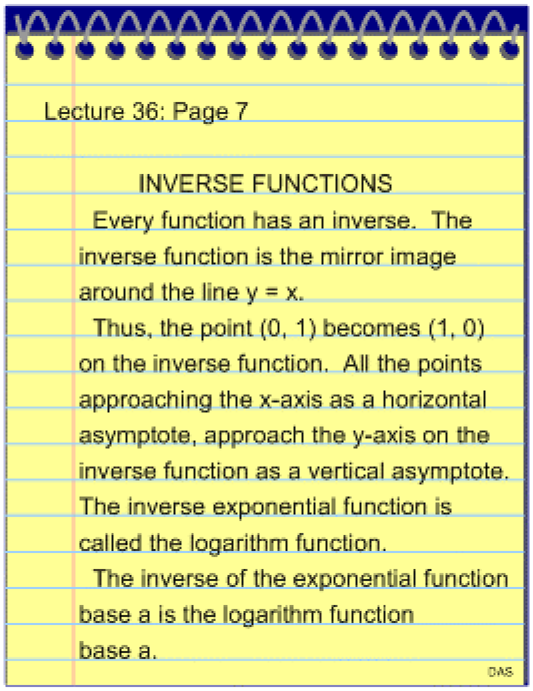
$a > 1$: Exponential growth

$0 < a < 1$: Exponential decay

Why wouldn't we look at an exponential function where $a = 1$? 1^x is always 1. Exponential functions can have bases greater than or less than one, but we only look at exponential functions with positive bases.

DAS

CEMP036-07



Lecture 36: Page 7

INVERSE FUNCTIONS

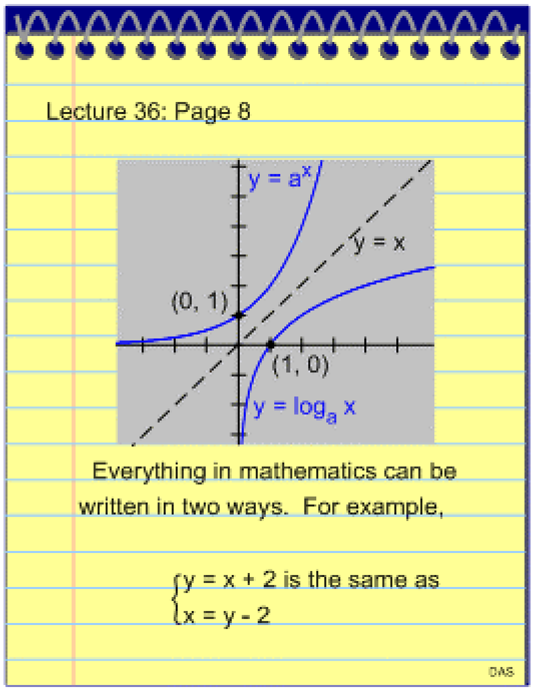
Every function has an inverse. The inverse function is the mirror image around the line $y = x$.

Thus, the point (0, 1) becomes (1, 0) on the inverse function. All the points approaching the x-axis as a horizontal asymptote, approach the y-axis on the inverse function as a vertical asymptote. The inverse exponential function is called the logarithm function.

The inverse of the exponential function base a is the logarithm function base a.

DAS

CEMP036-08



Lecture 36: Page 8

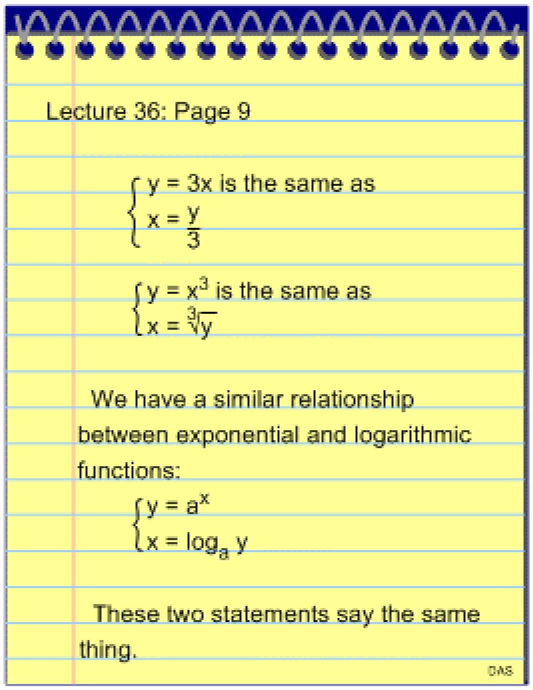
Everything in mathematics can be written in two ways. For example,

$$\begin{cases} y = x + 2 \text{ is the same as} \\ x = y - 2 \end{cases}$$

DAS

Lecture 36 Notes, Continued

CEMP036-09



Lecture 36: Page 9

$\begin{cases} y = 3x \text{ is the same as} \\ x = \frac{y}{3} \end{cases}$

$\begin{cases} y = x^3 \text{ is the same as} \\ x = \sqrt[3]{y} \end{cases}$

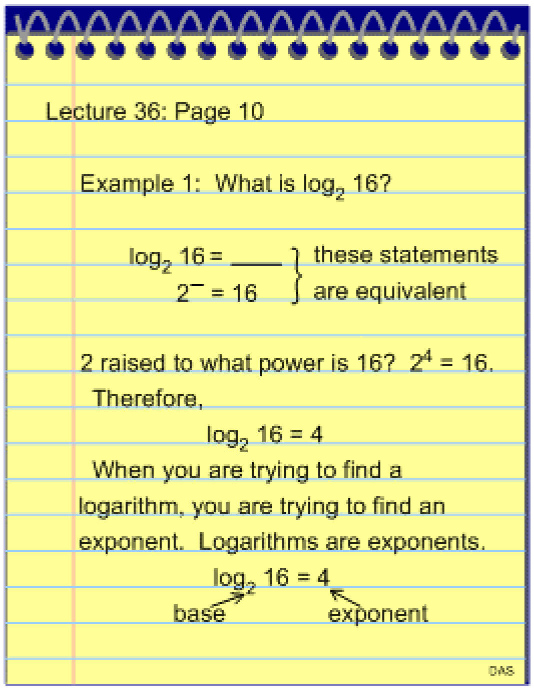
We have a similar relationship between exponential and logarithmic functions:

$\begin{cases} y = a^x \\ x = \log_a y \end{cases}$

These two statements say the same thing.

DAS

CEMP036-10



Lecture 36: Page 10

Example 1: What is $\log_2 16$?

$\left.\begin{array}{l} \log_2 16 = \underline{\quad\quad} \\ 2^{\_} = 16 \end{array}\right\}$ these statements are equivalent

2 raised to what power is 16? $2^4 = 16$.

Therefore,

$\log_2 16 = 4$

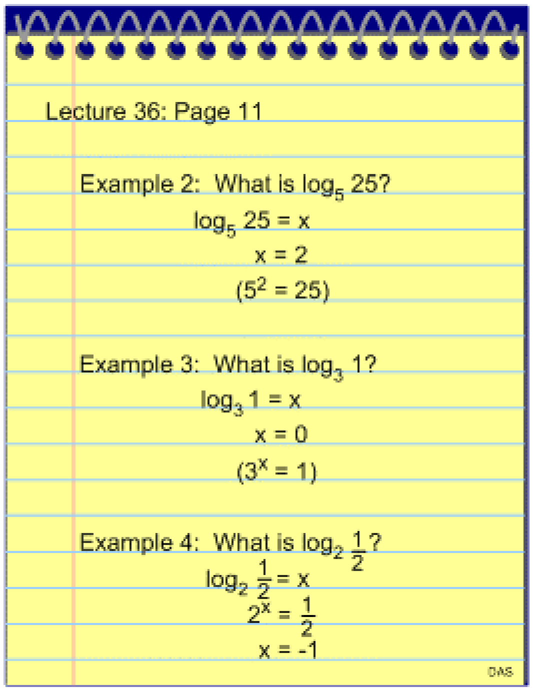
When you are trying to find a logarithm, you are trying to find an exponent. Logarithms are exponents.

$\log_2 16 = 4$

base exponent

DAS

CEMP036-11



Lecture 36: Page 11

Example 2: What is $\log_5 25$?

$\log_5 25 = x$

$x = 2$

$(5^2 = 25)$

Example 3: What is $\log_3 1$?

$\log_3 1 = x$

$x = 0$

$(3^x = 1)$

Example 4: What is $\log_2 \frac{1}{2}$?

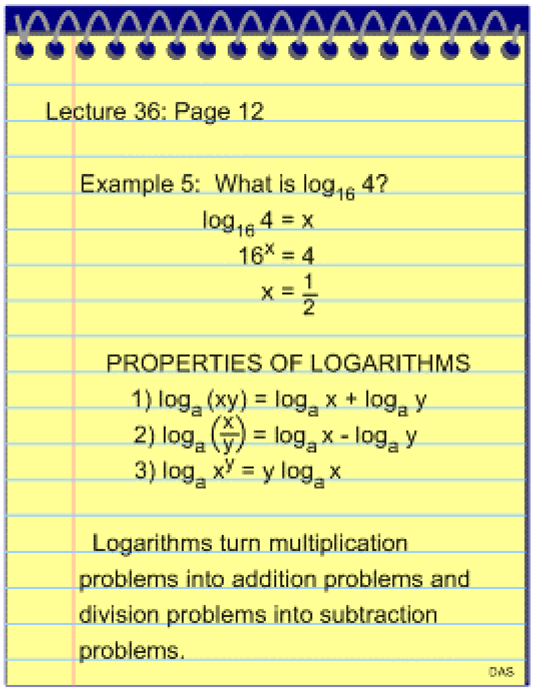
$\log_2 \frac{1}{2} = x$

$2^x = \frac{1}{2}$

$x = -1$

DAS

CEMP036-12



Lecture 36: Page 12

Example 5: What is $\log_{16} 4$?

$\log_{16} 4 = x$

$16^x = 4$

$x = \frac{1}{2}$

PROPERTIES OF LOGARITHMS

1) $\log_a (xy) = \log_a x + \log_a y$

2) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

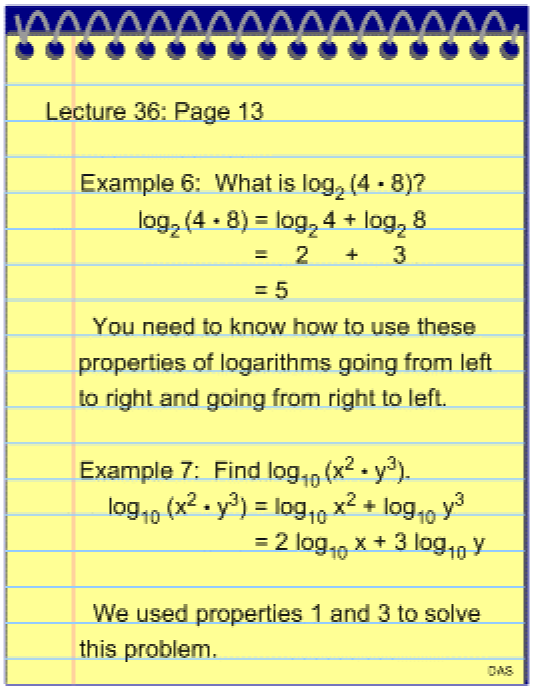
3) $\log_a x^y = y \log_a x$

Logarithms turn multiplication problems into addition problems and division problems into subtraction problems.

DAS

Lecture 36 Notes, Continued

CEMP036-13



Lecture 36: Page 13

Example 6: What is $\log_2 (4 \cdot 8)$?

$$\log_2 (4 \cdot 8) = \log_2 4 + \log_2 8$$
$$= 2 + 3$$
$$= 5$$

You need to know how to use these properties of logarithms going from left to right and going from right to left.

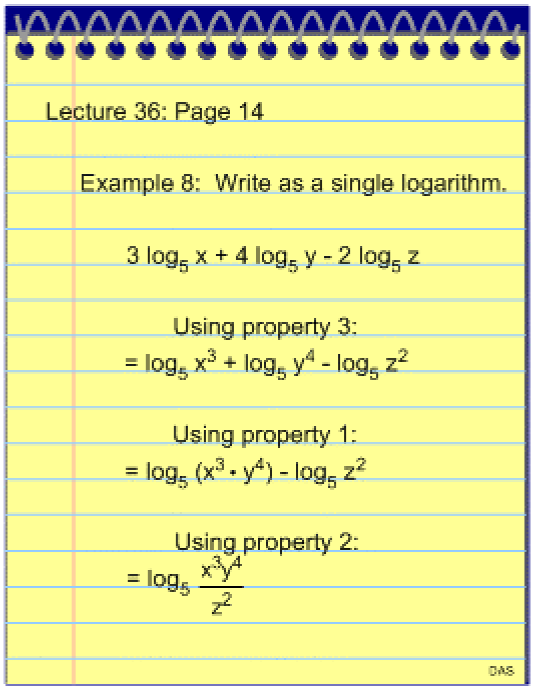
Example 7: Find $\log_{10} (x^2 \cdot y^3)$.

$$\log_{10} (x^2 \cdot y^3) = \log_{10} x^2 + \log_{10} y^3$$
$$= 2 \log_{10} x + 3 \log_{10} y$$

We used properties 1 and 3 to solve this problem.

DAS

CEMP036-14



Lecture 36: Page 14

Example 8: Write as a single logarithm.

$$3 \log_5 x + 4 \log_5 y - 2 \log_5 z$$

Using property 3:

$$= \log_5 x^3 + \log_5 y^4 - \log_5 z^2$$

Using property 1:

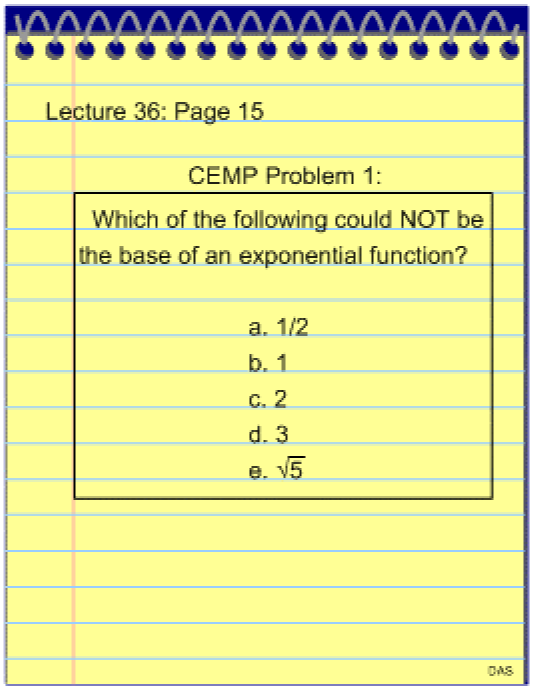
$$= \log_5 (x^3 \cdot y^4) - \log_5 z^2$$

Using property 2:

$$= \log_5 \frac{x^3y^4}{z^2}$$

DAS

CEMP036-15



Lecture 36: Page 15

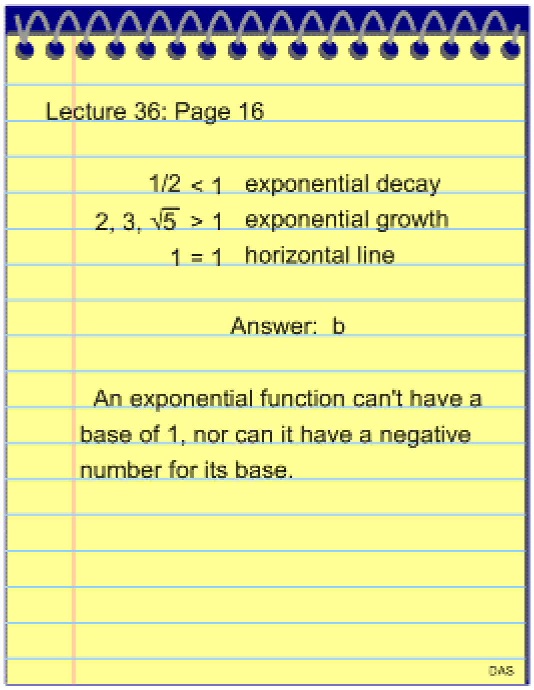
CEMP Problem 1:

Which of the following could NOT be the base of an exponential function?

a. 1/2
b. 1
c. 2
d. 3
e. $\sqrt{5}$

DAS

CEMP036-16



Lecture 36: Page 16

$1/2 < 1$	exponential decay
$2, 3, \sqrt{5} > 1$	exponential growth
$1 = 1$	horizontal line

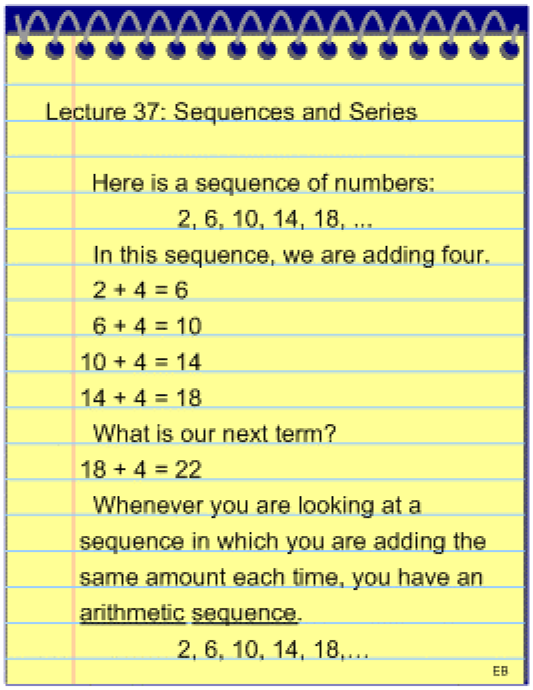
Answer: b

An exponential function can't have a base of 1, nor can it have a negative number for its base.

DAS

Lecture 37 Notes

CEMP037-01

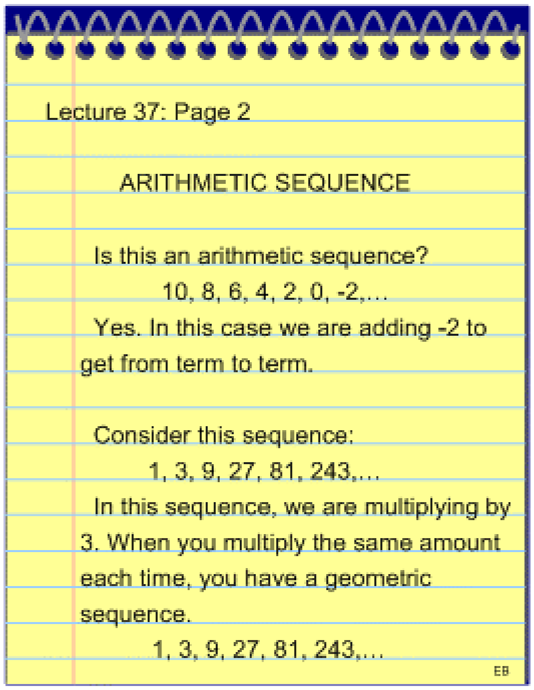


Lecture 37: Sequences and Series

Here is a sequence of numbers:
2, 6, 10, 14, 18, ...
In this sequence, we are adding four.
2 + 4 = 6
6 + 4 = 10
10 + 4 = 14
14 + 4 = 18
What is our next term?
18 + 4 = 22
Whenever you are looking at a sequence in which you are adding the same amount each time, you have an arithmetic sequence.
2, 6, 10, 14, 18,…

EB

CEMP037-02



Lecture 37: Page 2

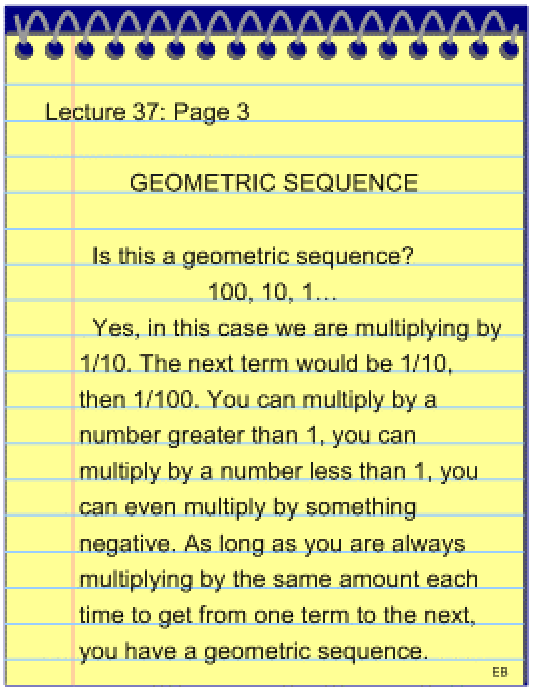
ARITHMETIC SEQUENCE

Is this an arithmetic sequence?
10, 8, 6, 4, 2, 0, -2,…
Yes. In this case we are adding -2 to get from term to term.

Consider this sequence:
1, 3, 9, 27, 81, 243,…
In this sequence, we are multiplying by 3. When you multiply the same amount each time, you have a geometric sequence.
1, 3, 9, 27, 81, 243,…

EB

CEMP037-03



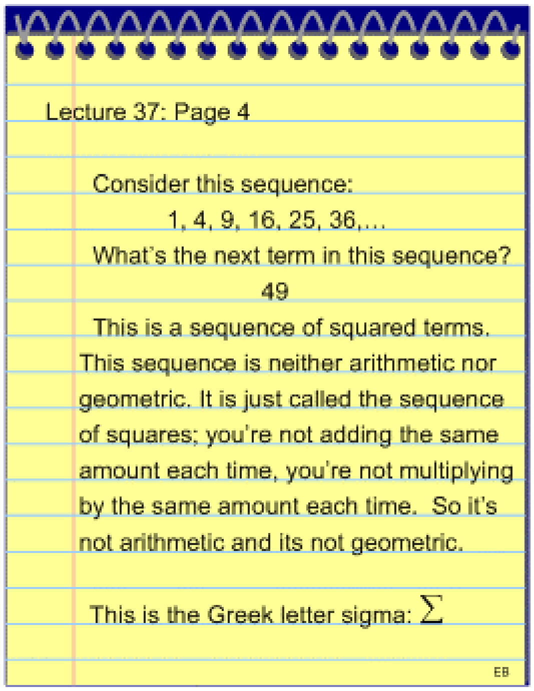
Lecture 37: Page 3

GEOMETRIC SEQUENCE

Is this a geometric sequence?
100, 10, 1…
Yes, in this case we are multiplying by 1/10. The next term would be 1/10, then 1/100. You can multiply by a number greater than 1, you can multiply by a number less than 1, you can even multiply by something negative. As long as you are always multiplying by the same amount each time to get from one term to the next, you have a geometric sequence.

EB

CEMP037-04



Lecture 37: Page 4

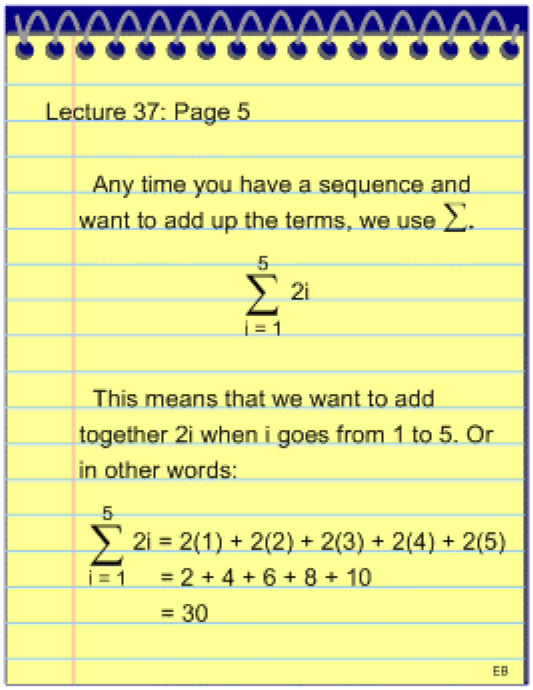
Consider this sequence:
1, 4, 9, 16, 25, 36,…
What's the next term in this sequence?
49
This is a sequence of squared terms. This sequence is neither arithmetic nor geometric. It is just called the sequence of squares; you're not adding the same amount each time, you're not multiplying by the same amount each time. So it's not arithmetic and its not geometric.

This is the Greek letter sigma: Σ

EB

Lecture 37 Notes, Continued

CEMP037-05



Lecture 37: Page 5

Any time you have a sequence and want to add up the terms, we use $\sum$.

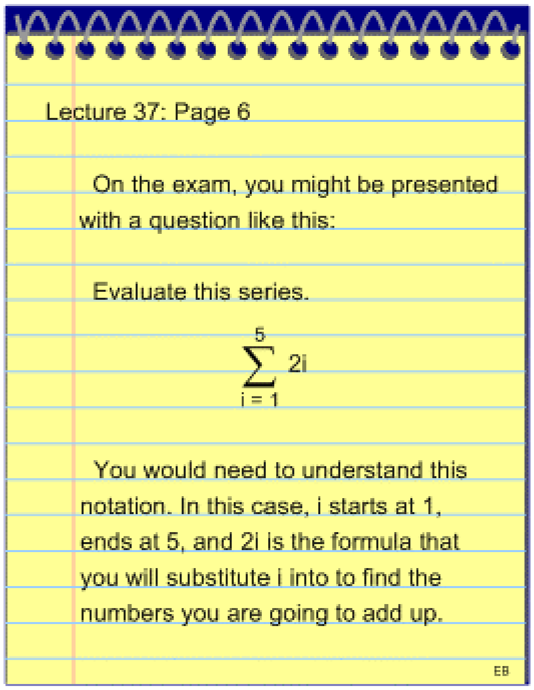
$$\sum_{i=1}^{5} 2i$$

This means that we want to add together 2i when i goes from 1 to 5. Or in other words:

$$\begin{aligned}\sum_{i=1}^{5} 2i &= 2(1) + 2(2) + 2(3) + 2(4) + 2(5)\\ &= 2 + 4 + 6 + 8 + 10\\ &= 30\end{aligned}$$

EB

CEMP037-06



Lecture 37: Page 6

On the exam, you might be presented with a question like this:

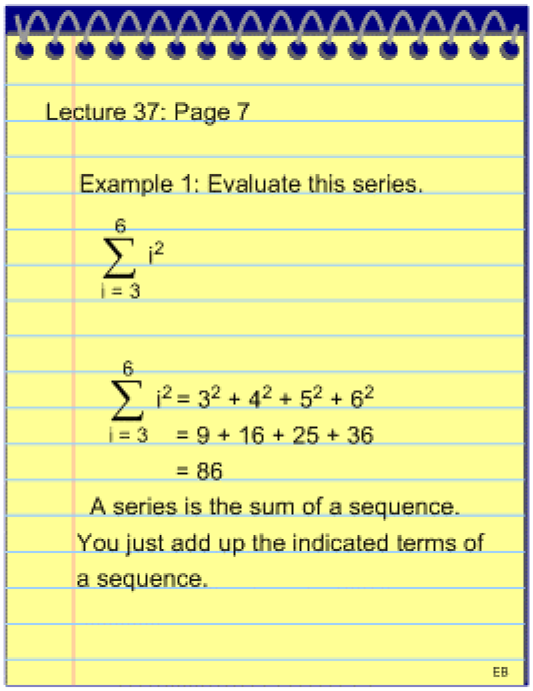
Evaluate this series.

$$\sum_{i=1}^{5} 2i$$

You would need to understand this notation. In this case, i starts at 1, ends at 5, and 2i is the formula that you will substitute i into to find the numbers you are going to add up.

EB

CEMP037-07



Lecture 37: Page 7

Example 1: Evaluate this series.

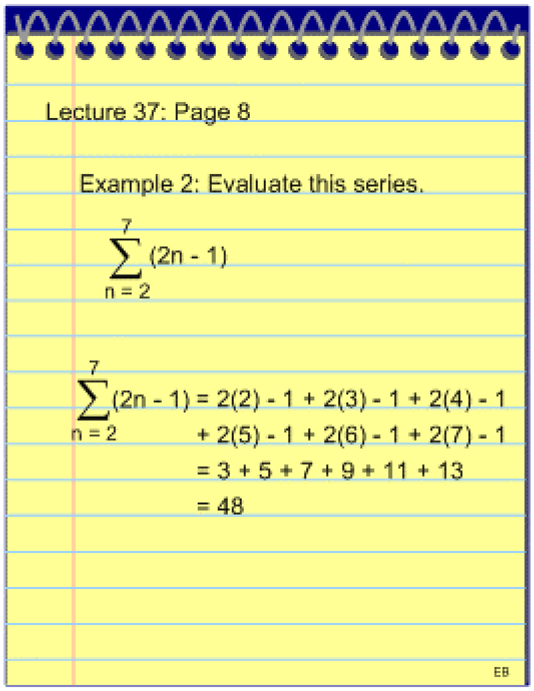
$$\sum_{i=3}^{6} i^2$$

$$\begin{aligned}\sum_{i=3}^{6} i^2 &= 3^2 + 4^2 + 5^2 + 6^2\\ &= 9 + 16 + 25 + 36\\ &= 86\end{aligned}$$

A series is the sum of a sequence. You just add up the indicated terms of a sequence.

EB

CEMP037-08



Lecture 37: Page 8

Example 2: Evaluate this series.

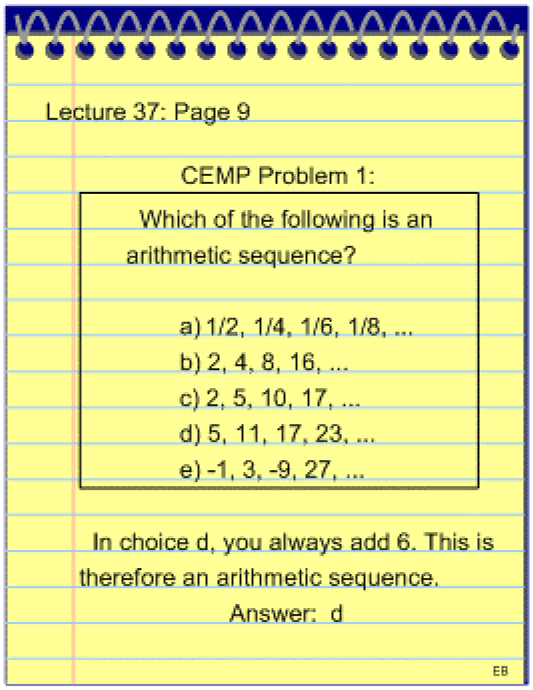
$$\sum_{n=2}^{7} (2n - 1)$$

$$\begin{aligned}\sum_{n=2}^{7} (2n - 1) &= 2(2) - 1 + 2(3) - 1 + 2(4) - 1\\ &\quad + 2(5) - 1 + 2(6) - 1 + 2(7) - 1\\ &= 3 + 5 + 7 + 9 + 11 + 13\\ &= 48\end{aligned}$$

EB

Lecture 37 Notes, Continued

CEMP037-09



Lecture 37: Page 9

CEMP Problem 1:

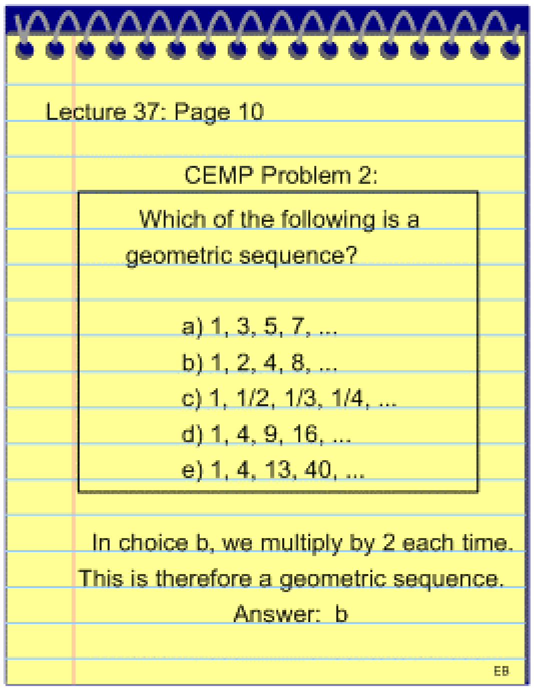
Which of the following is an arithmetic sequence?

a) 1/2, 1/4, 1/6, 1/8, ...
b) 2, 4, 8, 16, ...
c) 2, 5, 10, 17, ...
d) 5, 11, 17, 23, ...
e) -1, 3, -9, 27, ...

In choice d, you always add 6. This is therefore an arithmetic sequence.

Answer: d

CEMP037-10



Lecture 37: Page 10

CEMP Problem 2:

Which of the following is a geometric sequence?

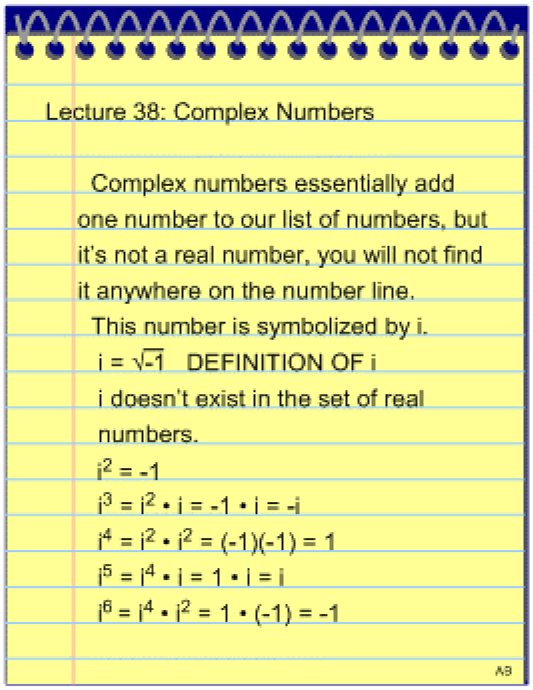
a) 1, 3, 5, 7, ...
b) 1, 2, 4, 8, ...
c) 1, 1/2, 1/3, 1/4, ...
d) 1, 4, 9, 16, ...
e) 1, 4, 13, 40, ...

In choice b, we multiply by 2 each time. This is therefore a geometric sequence.

Answer: b

Lecture 38 Notes

CEMP038-01



Lecture 38: Complex Numbers

Complex numbers essentially add one number to our list of numbers, but it's not a real number, you will not find it anywhere on the number line.

This number is symbolized by i.

$i = \sqrt{-1}$ DEFINITION OF i

i doesn't exist in the set of real numbers.

$i^2 = -1$

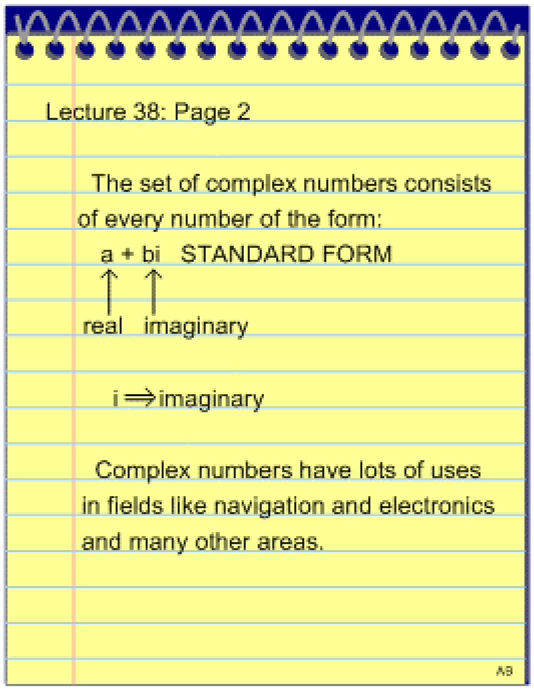
$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

CEMP038-02



Lecture 38: Page 2

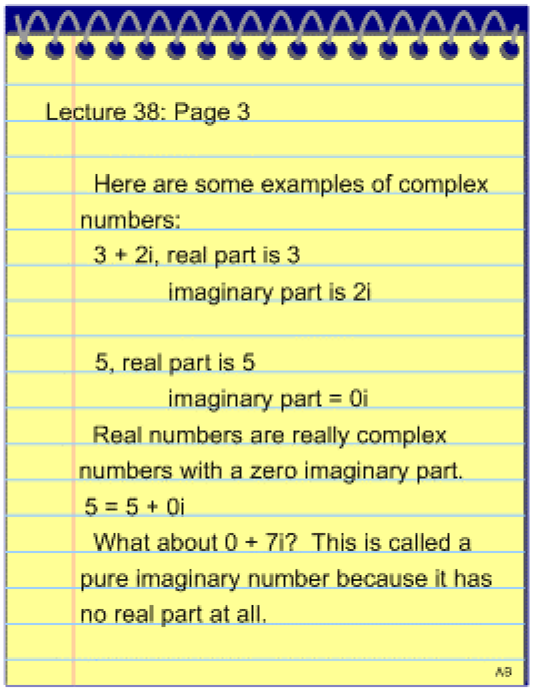
The set of complex numbers consists of every number of the form:

$$\underset{\text{real}}{\overset{\uparrow}{a}} + \underset{\text{imaginary}}{\overset{\uparrow}{b}}i \quad \text{STANDARD FORM}$$

$i \Rightarrow$ imaginary

Complex numbers have lots of uses in fields like navigation and electronics and many other areas.

CEMP038-03



Lecture 38: Page 3

Here are some examples of complex numbers:

3 + 2i, real part is 3
imaginary part is 2i

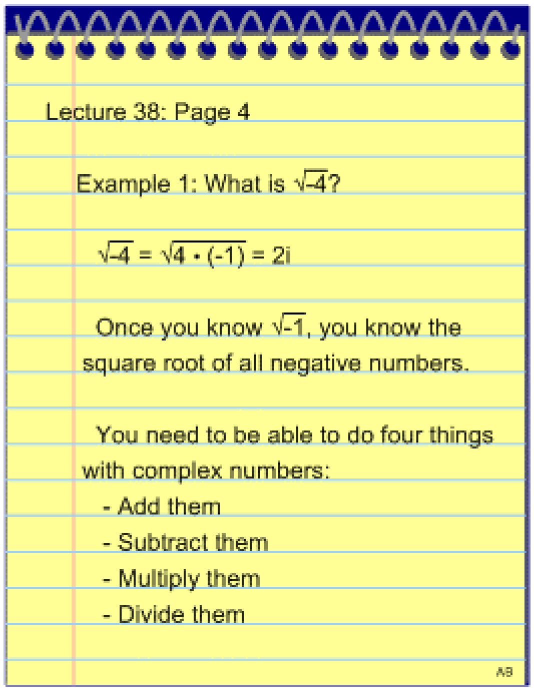
5, real part is 5
imaginary part = 0i

Real numbers are really complex numbers with a zero imaginary part.

5 = 5 + 0i

What about 0 + 7i? This is called a pure imaginary number because it has no real part at all.

CEMP038-04



Lecture 38: Page 4

Example 1: What is $\sqrt{-4}$?

$\sqrt{-4} = \sqrt{4 \cdot (-1)} = 2i$

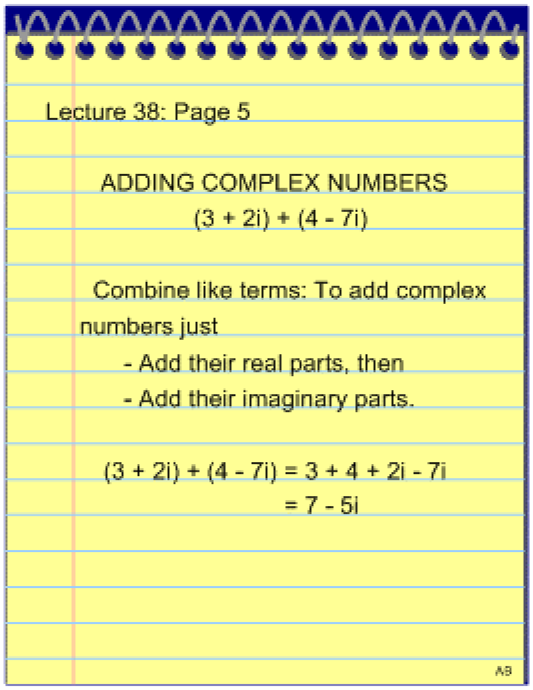
Once you know $\sqrt{-1}$, you know the square root of all negative numbers.

You need to be able to do four things with complex numbers:

- Add them
- Subtract them
- Multiply them
- Divide them

Lecture 38 Notes, Continued

CEMP038-05



Lecture 38: Page 5

ADDING COMPLEX NUMBERS

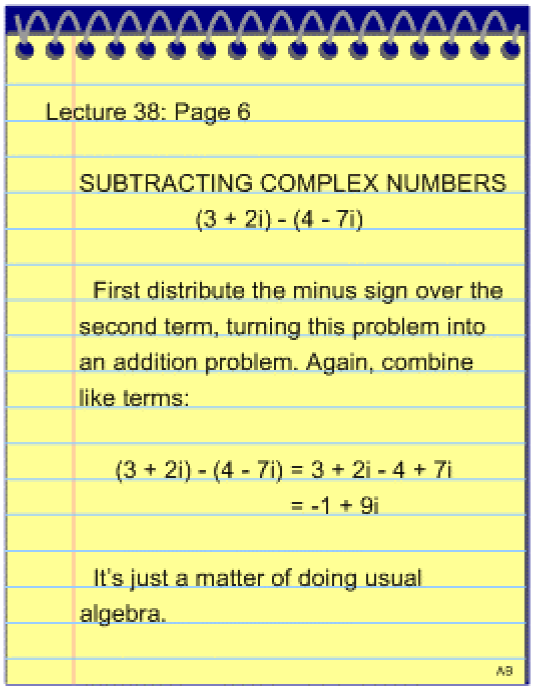
$(3 + 2i) + (4 - 7i)$

Combine like terms: To add complex numbers just

- Add their real parts, then
- Add their imaginary parts.

$$(3 + 2i) + (4 - 7i) = 3 + 4 + 2i - 7i$$
$$= 7 - 5i$$

CEMP038-06



Lecture 38: Page 6

SUBTRACTING COMPLEX NUMBERS

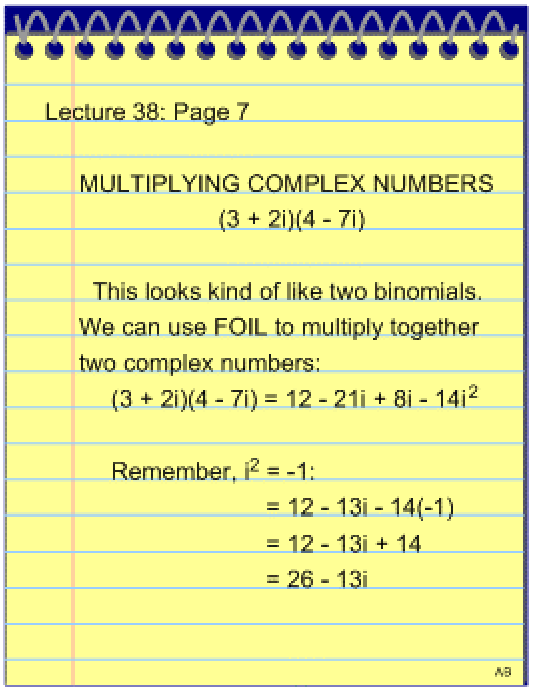
$(3 + 2i) - (4 - 7i)$

First distribute the minus sign over the second term, turning this problem into an addition problem. Again, combine like terms:

$$(3 + 2i) - (4 - 7i) = 3 + 2i - 4 + 7i$$
$$= -1 + 9i$$

It's just a matter of doing usual algebra.

CEMP038-07



Lecture 38: Page 7

MULTIPLYING COMPLEX NUMBERS

$(3 + 2i)(4 - 7i)$

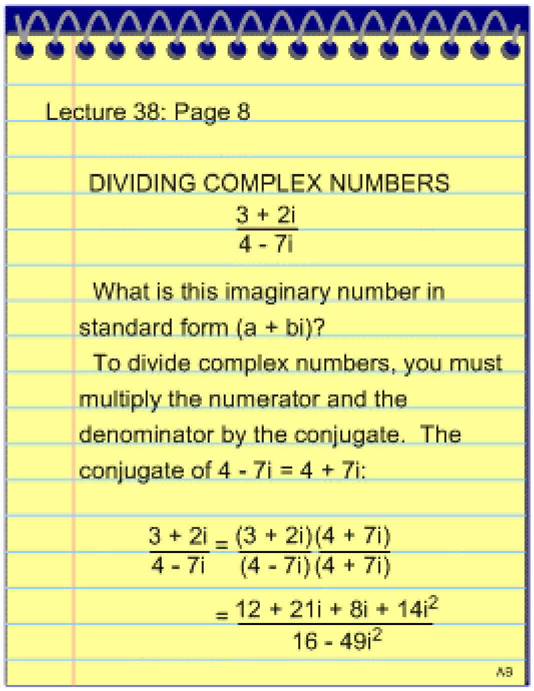
This looks kind of like two binomials. We can use FOIL to multiply together two complex numbers:

$$(3 + 2i)(4 - 7i) = 12 - 21i + 8i - 14i^2$$

Remember, $i^2 = -1$:

$$= 12 - 13i - 14(-1)$$
$$= 12 - 13i + 14$$
$$= 26 - 13i$$

CEMP038-08



Lecture 38: Page 8

DIVIDING COMPLEX NUMBERS

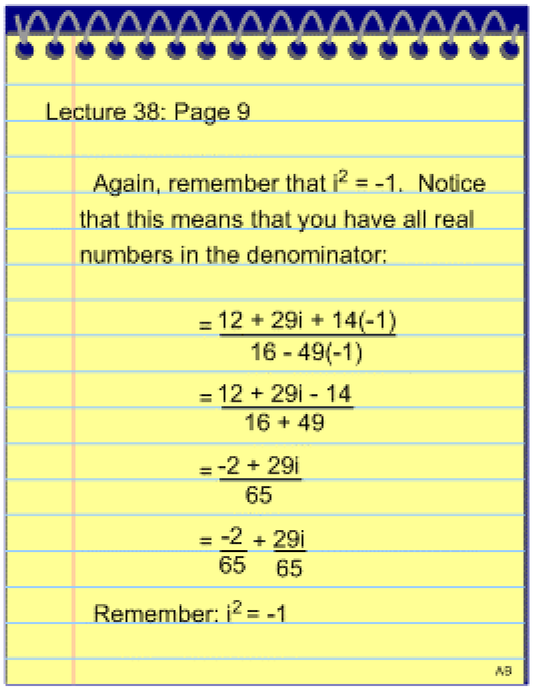
$$\frac{3 + 2i}{4 - 7i}$$

What is this imaginary number in standard form $(a + bi)$?

To divide complex numbers, you must multiply the numerator and the denominator by the conjugate. The conjugate of $4 - 7i = 4 + 7i$:

$$\frac{3 + 2i}{4 - 7i} = \frac{(3 + 2i)(4 + 7i)}{(4 - 7i)(4 + 7i)}$$
$$= \frac{12 + 21i + 8i + 14i^2}{16 - 49i^2}$$

CEMP038-09



Lecture 38: Page 9

Again, remember that $i^2 = -1$. Notice that this means that you have all real numbers in the denominator:

$$= \frac{12 + 29i + 14(-1)}{16 - 49(-1)}$$

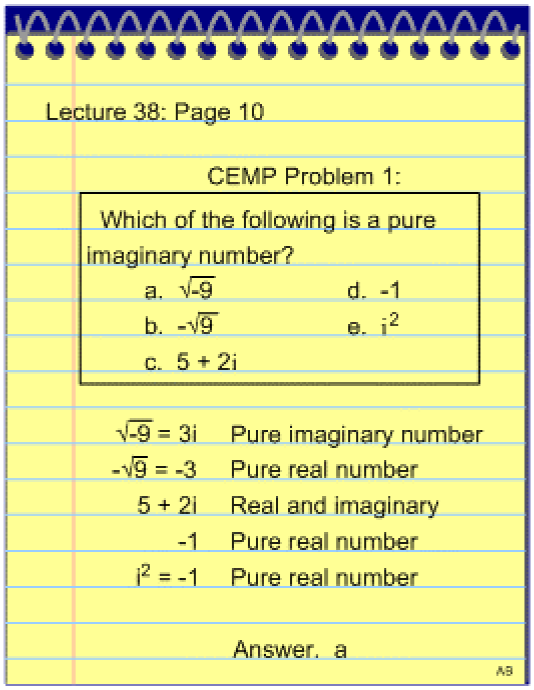
$$= \frac{12 + 29i - 14}{16 + 49}$$

$$= \frac{-2 + 29i}{65}$$

$$= \frac{-2}{65} + \frac{29i}{65}$$

Remember: $i^2 = -1$

CEMP038-10



Lecture 38: Page 10

CEMP Problem 1:

Which of the following is a pure imaginary number?

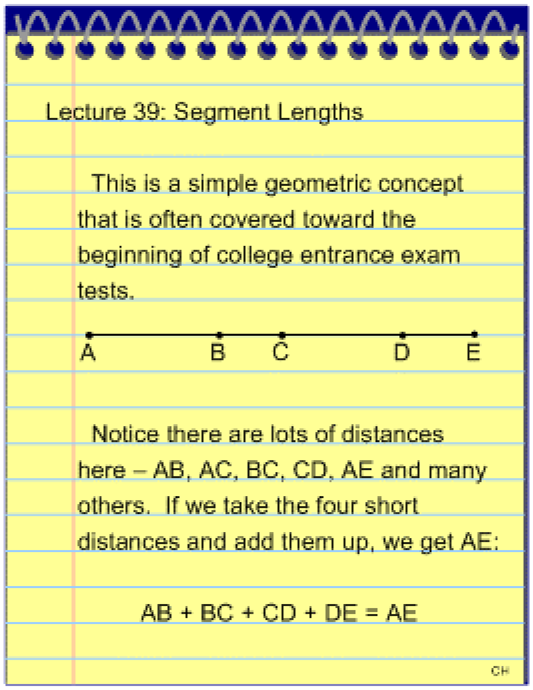
a. $\sqrt{-9}$
b. $-\sqrt{9}$
c. $5 + 2i$
d. -1
e. i^2

$\sqrt{-9} = 3i$	Pure imaginary number
$-\sqrt{9} = -3$	Pure real number
$5 + 2i$	Real and imaginary
-1	Pure real number
$i^2 = -1$	Pure real number

Answer. a

Lecture 39 Notes

CEMP039-01



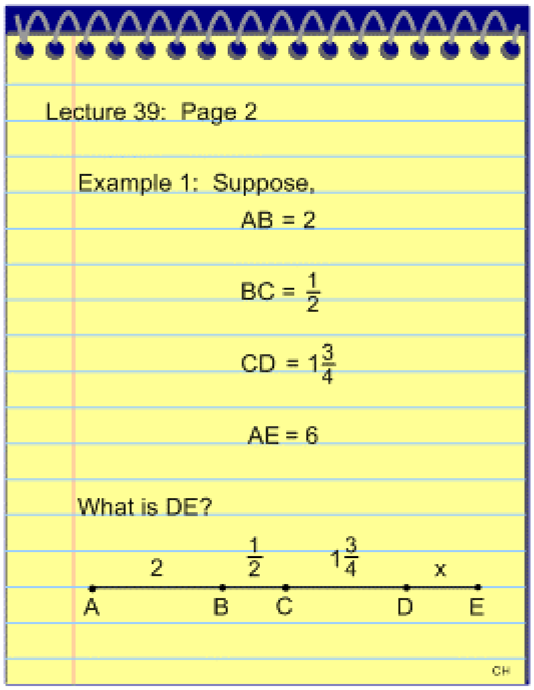
Lecture 39: Segment Lengths

This is a simple geometric concept that is often covered toward the beginning of college entrance exam tests.

Notice there are lots of distances here – AB, AC, BC, CD, AE and many others. If we take the four short distances and add them up, we get AE:

AB + BC + CD + DE = AE

CEMP039-02



Lecture 39: Page 2

Example 1: Suppose,

$AB = 2$

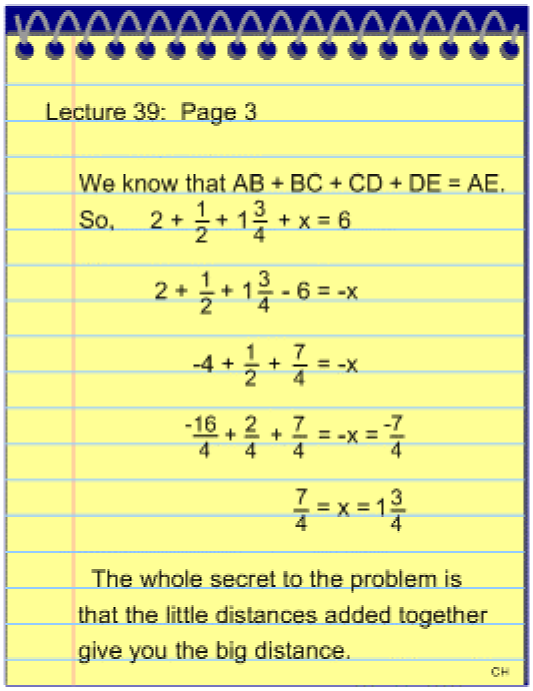
$BC = \frac{1}{2}$

$CD = 1\frac{3}{4}$

$AE = 6$

What is DE?

CEMP039-03



Lecture 39: Page 3

We know that AB + BC + CD + DE = AE.

So, $2 + \frac{1}{2} + 1\frac{3}{4} + x = 6$

$$2 + \frac{1}{2} + 1\frac{3}{4} - 6 = -x$$

$$-4 + \frac{1}{2} + \frac{7}{4} = -x$$

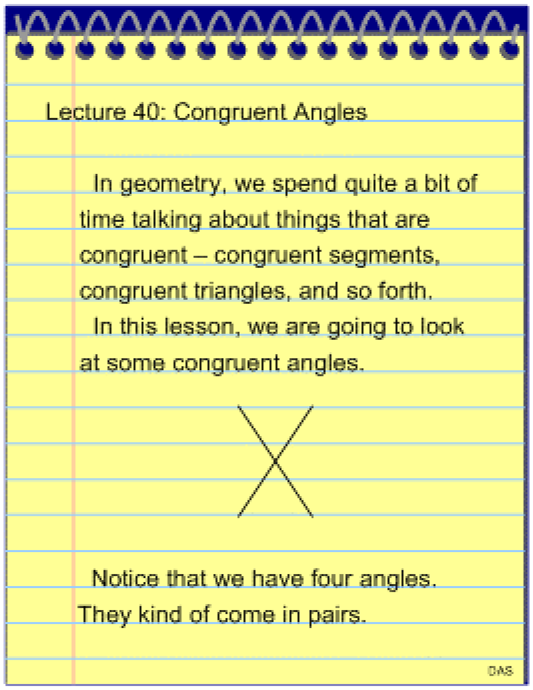
$$\frac{-16}{4} + \frac{2}{4} + \frac{7}{4} = -x = \frac{-7}{4}$$

$$\frac{7}{4} = x = 1\frac{3}{4}$$

The whole secret to the problem is that the little distances added together give you the big distance.

Lecture 40 Notes

CEMP040-01



Lecture 40: Congruent Angles

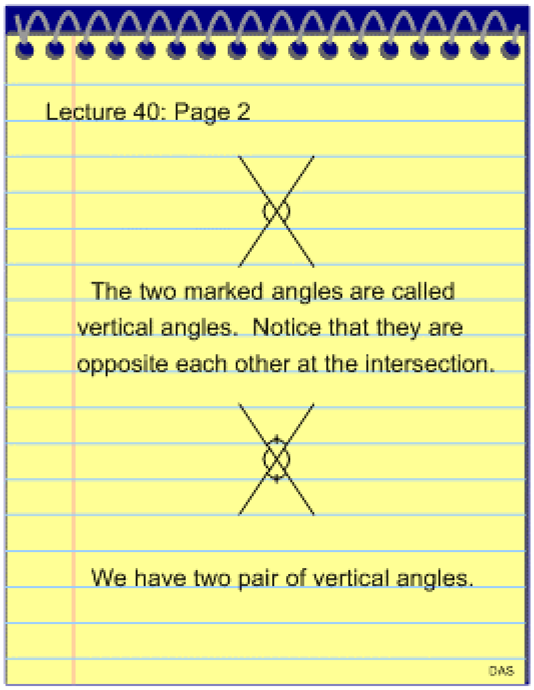
In geometry, we spend quite a bit of time talking about things that are congruent – congruent segments, congruent triangles, and so forth.

In this lesson, we are going to look at some congruent angles.

Notice that we have four angles. They kind of come in pairs.

DAS

CEMP040-02



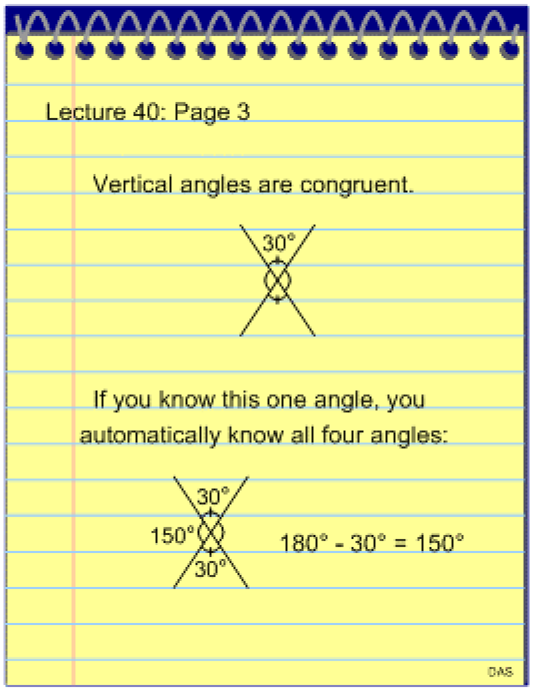
Lecture 40: Page 2

The two marked angles are called vertical angles. Notice that they are opposite each other at the intersection.

We have two pair of vertical angles.

DAS

CEMP040-03



Lecture 40: Page 3

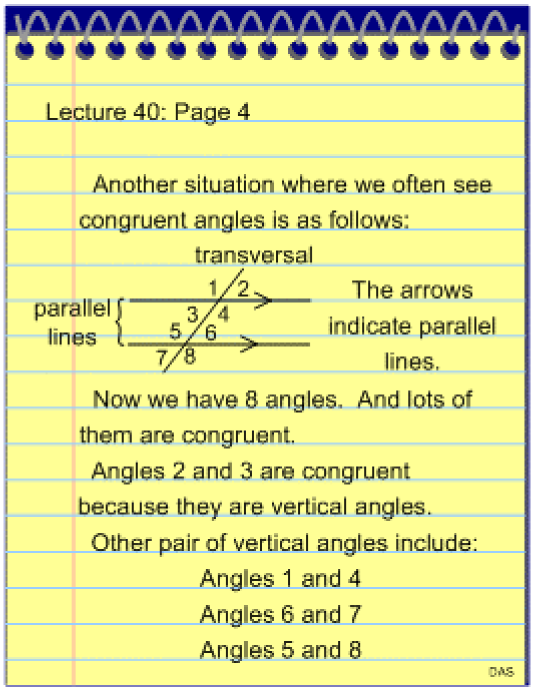
Vertical angles are congruent.

If you know this one angle, you automatically know all four angles:

180° - 30° = 150°

DAS

CEMP040-04



Lecture 40: Page 4

Another situation where we often see congruent angles is as follows:

The arrows indicate parallel lines.

Now we have 8 angles. And lots of them are congruent.

Angles 2 and 3 are congruent because they are vertical angles.

Other pair of vertical angles include:

Angles 1 and 4

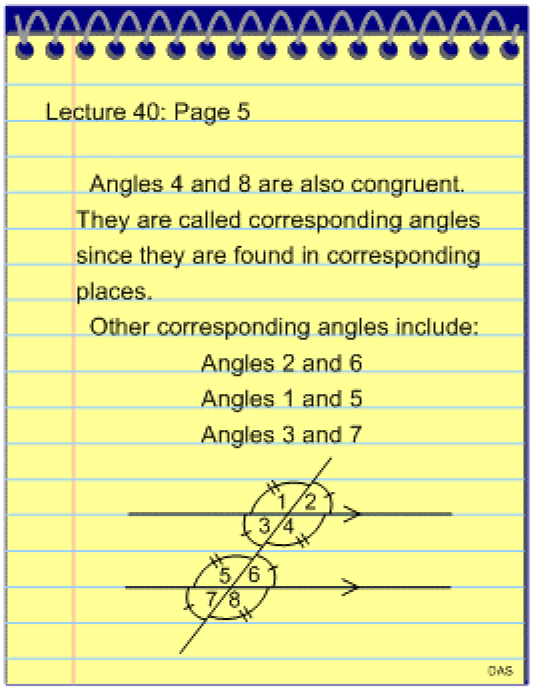
Angles 6 and 7

Angles 5 and 8

DAS

Lecture 40 Notes, Continued

CEMP040-05



Lecture 40: Page 5

Angles 4 and 8 are also congruent. They are called corresponding angles since they are found in corresponding places.

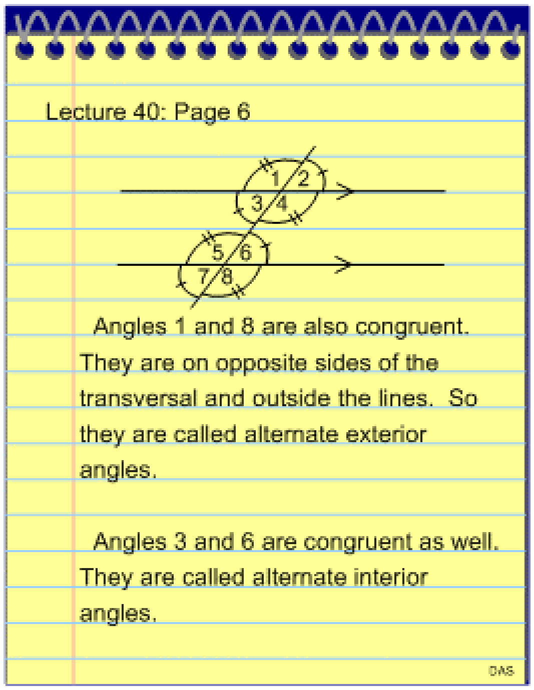
Other corresponding angles include:

Angles 2 and 6

Angles 1 and 5

Angles 3 and 7

CEMP040-06

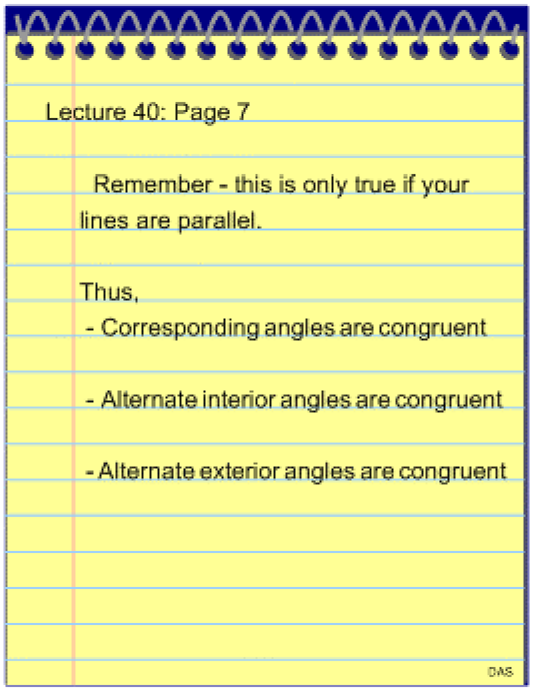


Lecture 40: Page 6

Angles 1 and 8 are also congruent. They are on opposite sides of the transversal and outside the lines. So they are called alternate exterior angles.

Angles 3 and 6 are congruent as well. They are called alternate interior angles.

CEMP040-07



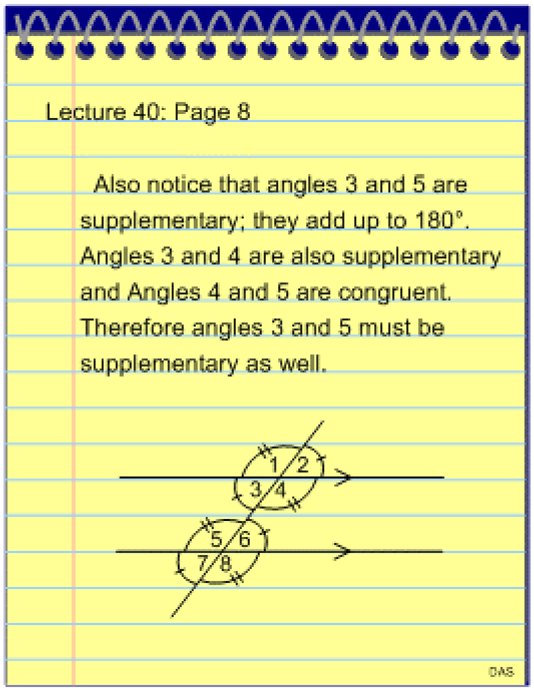
Lecture 40: Page 7

Remember - this is only true if your lines are parallel.

Thus,

- Corresponding angles are congruent
- Alternate interior angles are congruent
- Alternate exterior angles are congruent

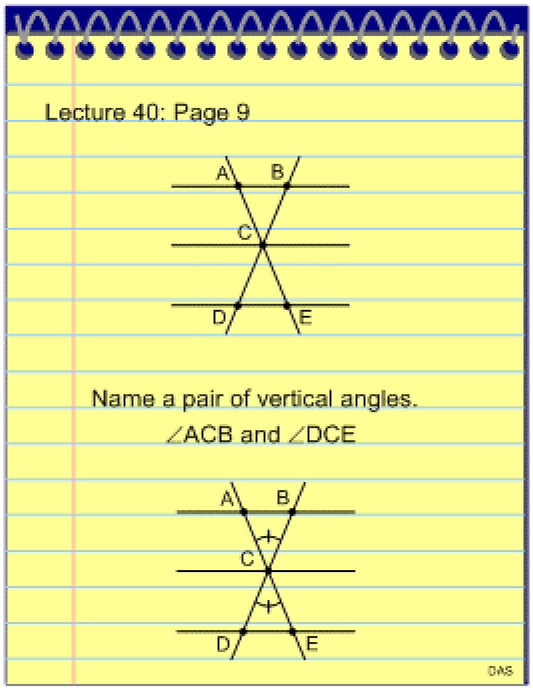
CEMP040-08



Lecture 40: Page 8

Also notice that angles 3 and 5 are supplementary; they add up to 180°. Angles 3 and 4 are also supplementary and Angles 4 and 5 are congruent. Therefore angles 3 and 5 must be supplementary as well.

CEMP040-09



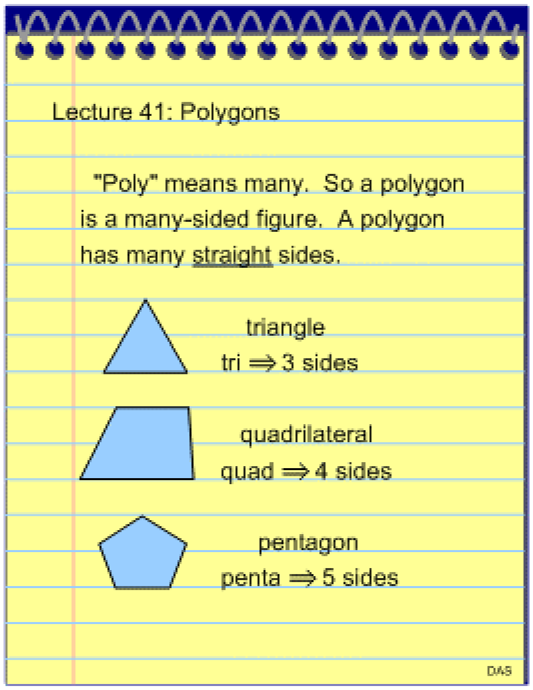
Lecture 40: Page 9

Name a pair of vertical angles.

∠ACB and ∠DCE

Lecture 41 Notes

CEMP041-01



Lecture 41: Polygons

"Poly" means many. So a polygon is a many-sided figure. A polygon has many straight sides.

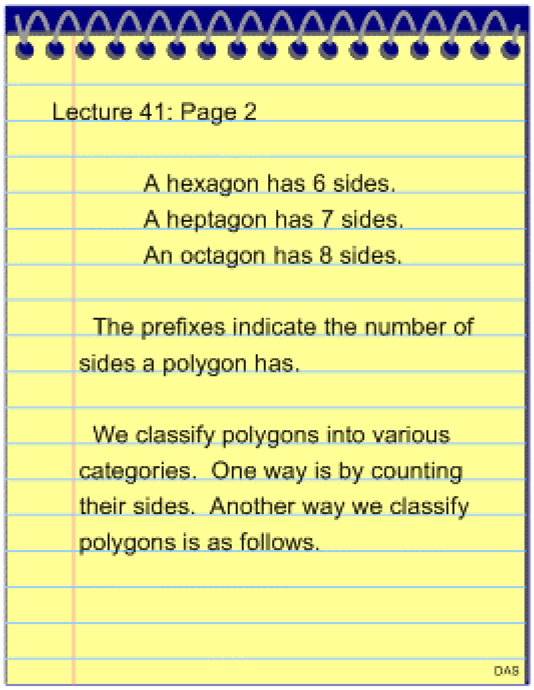
triangle
tri ⇒ 3 sides

quadrilateral
quad ⇒ 4 sides

pentagon
penta ⇒ 5 sides

DAS

CEMP041-02



Lecture 41: Page 2

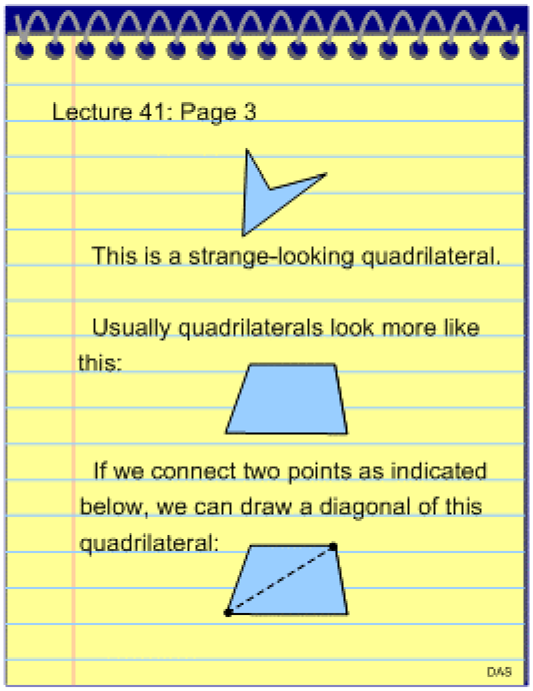
A hexagon has 6 sides.
A heptagon has 7 sides.
An octagon has 8 sides.

The prefixes indicate the number of sides a polygon has.

We classify polygons into various categories. One way is by counting their sides. Another way we classify polygons is as follows.

DAS

CEMP041-03



Lecture 41: Page 3

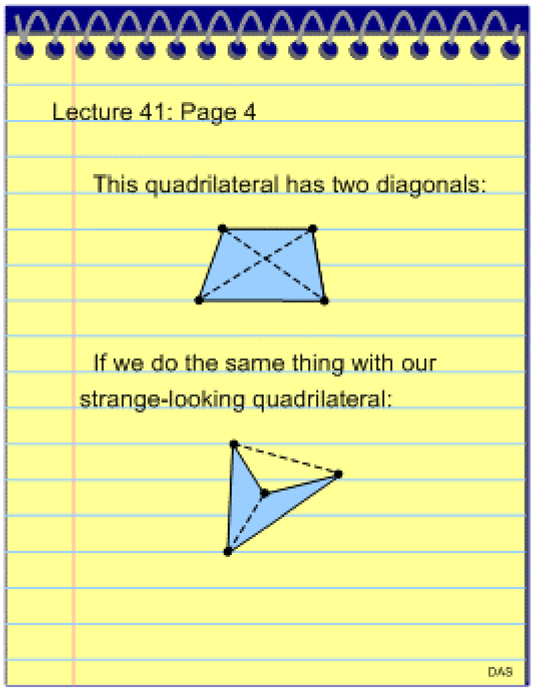
This is a strange-looking quadrilateral.

Usually quadrilaterals look more like this:

If we connect two points as indicated below, we can draw a diagonal of this quadrilateral:

DAS

CEMP041-04



Lecture 41: Page 4

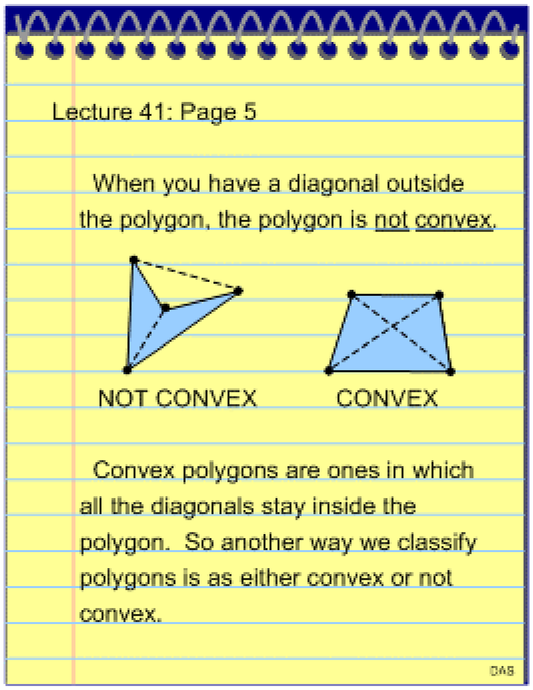
This quadrilateral has two diagonals:

If we do the same thing with our strange-looking quadrilateral:

DAS

Lecture 41 Notes, Continued

CEMP041-05



Lecture 41: Page 5

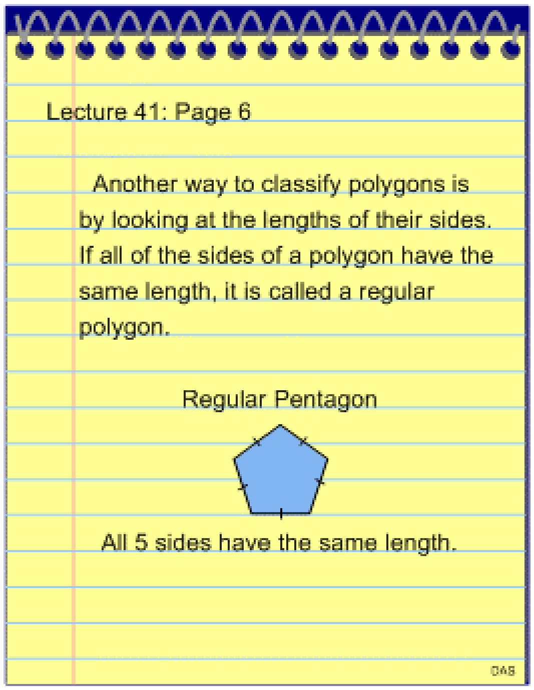
When you have a diagonal outside the polygon, the polygon is not convex.

NOT CONVEX

CONVEX

Convex polygons are ones in which all the diagonals stay inside the polygon. So another way we classify polygons is as either convex or not convex.

CEMP041-06



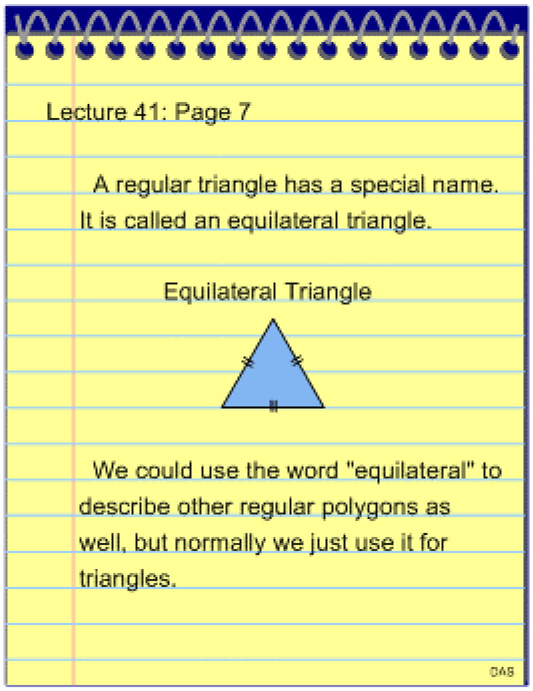
Lecture 41: Page 6

Another way to classify polygons is by looking at the lengths of their sides. If all of the sides of a polygon have the same length, it is called a regular polygon.

Regular Pentagon

All 5 sides have the same length.

CEMP041-07



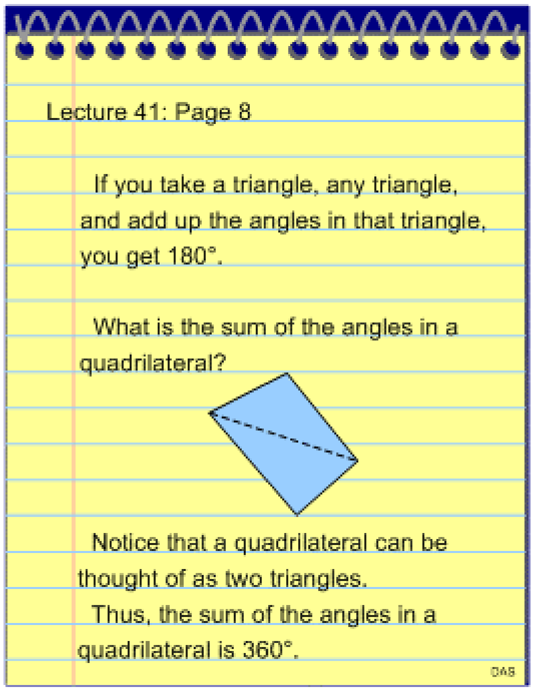
Lecture 41: Page 7

A regular triangle has a special name. It is called an equilateral triangle.

Equilateral Triangle

We could use the word "equilateral" to describe other regular polygons as well, but normally we just use it for triangles.

CEMP041-08



Lecture 41: Page 8

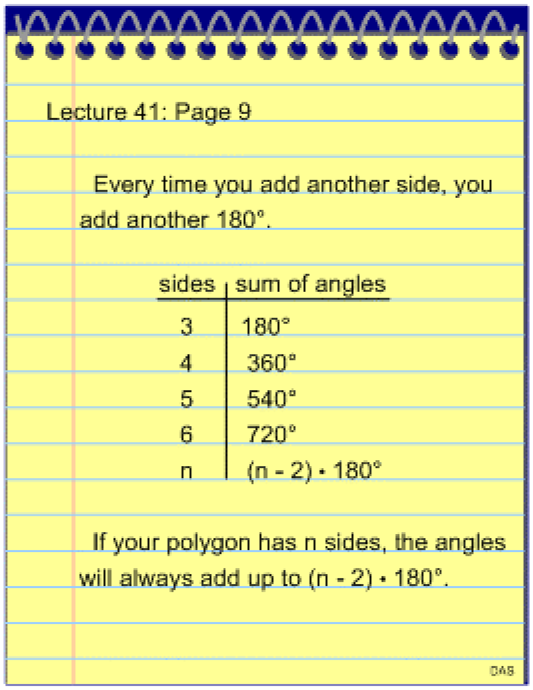
If you take a triangle, any triangle, and add up the angles in that triangle, you get 180°.

What is the sum of the angles in a quadrilateral?

Notice that a quadrilateral can be thought of as two triangles. Thus, the sum of the angles in a quadrilateral is 360°.

Lecture 41 Notes, Continued

CEMP041-09



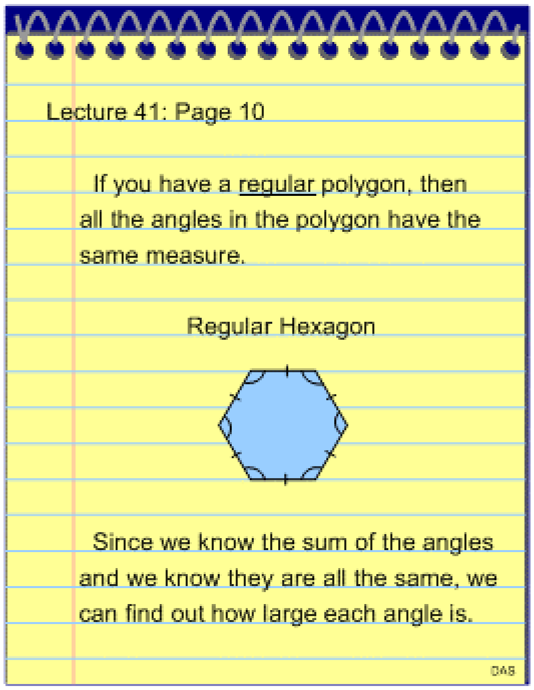
Lecture 41: Page 9

Every time you add another side, you add another 180°.

sides	sum of angles
3	180°
4	360°
5	540°
6	720°
n	$(n - 2) \cdot 180^\circ$

If your polygon has n sides, the angles will always add up to $(n - 2) \cdot 180^\circ$.

CEMP041-10



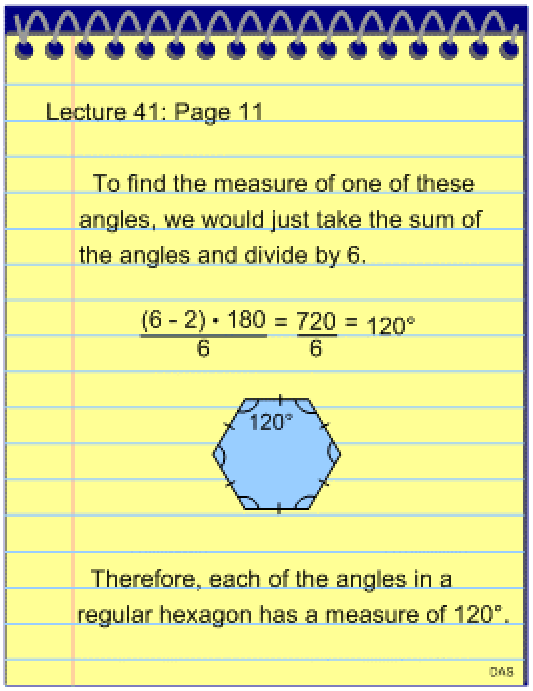
Lecture 41: Page 10

If you have a regular polygon, then all the angles in the polygon have the same measure.

Regular Hexagon

Since we know the sum of the angles and we know they are all the same, we can find out how large each angle is.

CEMP041-11



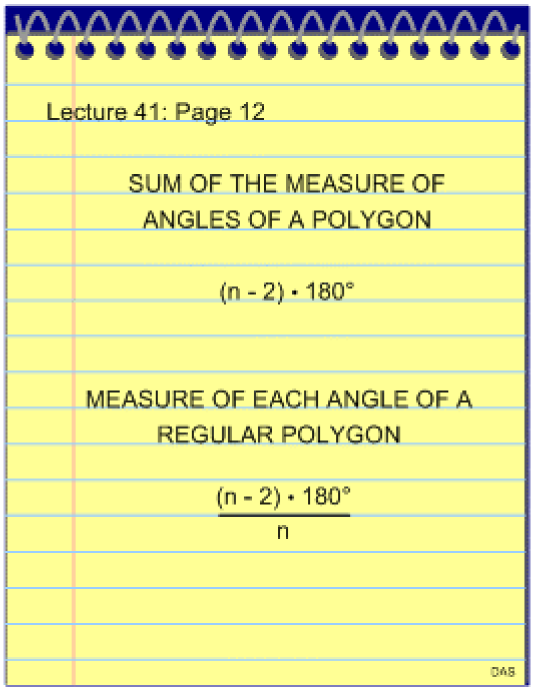
Lecture 41: Page 11

To find the measure of one of these angles, we would just take the sum of the angles and divide by 6.

$$\frac{(6 - 2) \cdot 180}{6} = \frac{720}{6} = 120^\circ$$

Therefore, each of the angles in a regular hexagon has a measure of 120°.

CEMP041-12



Lecture 41: Page 12

SUM OF THE MEASURE OF ANGLES OF A POLYGON

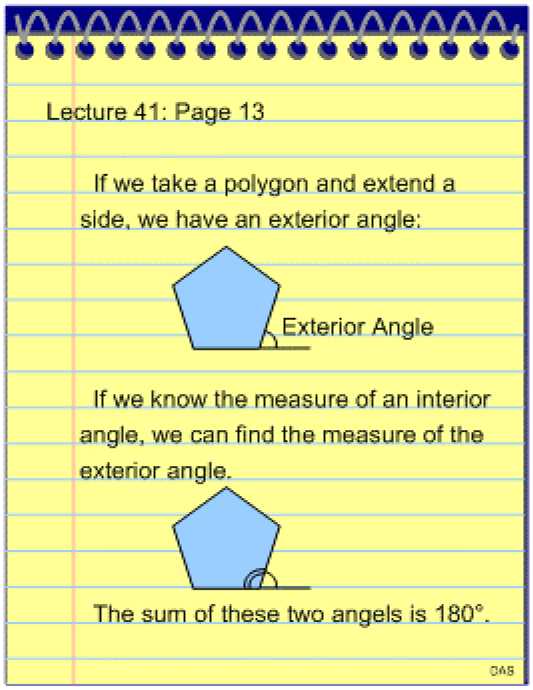
$$(n - 2) \cdot 180^\circ$$

MEASURE OF EACH ANGLE OF A REGULAR POLYGON

$$\frac{(n - 2) \cdot 180^\circ}{n}$$

Lecture 41 Notes, Continued

CEMP041-13



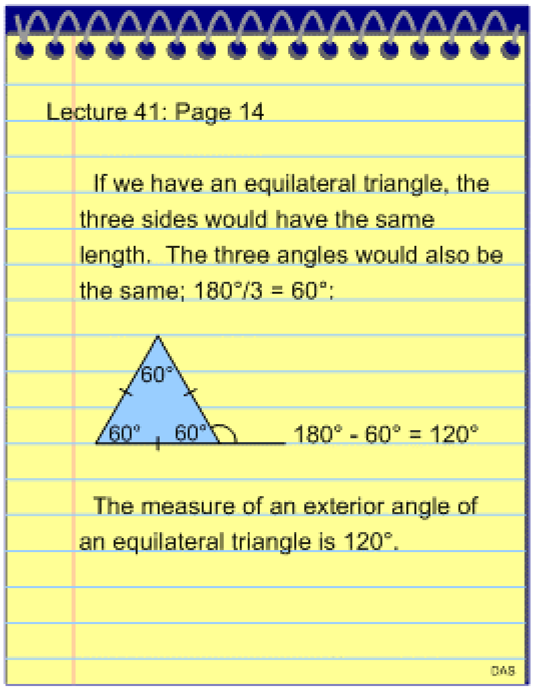
Lecture 41: Page 13

If we take a polygon and extend a side, we have an exterior angle:

If we know the measure of an interior angle, we can find the measure of the exterior angle.

The sum of these two angels is 180°.

CEMP041-14



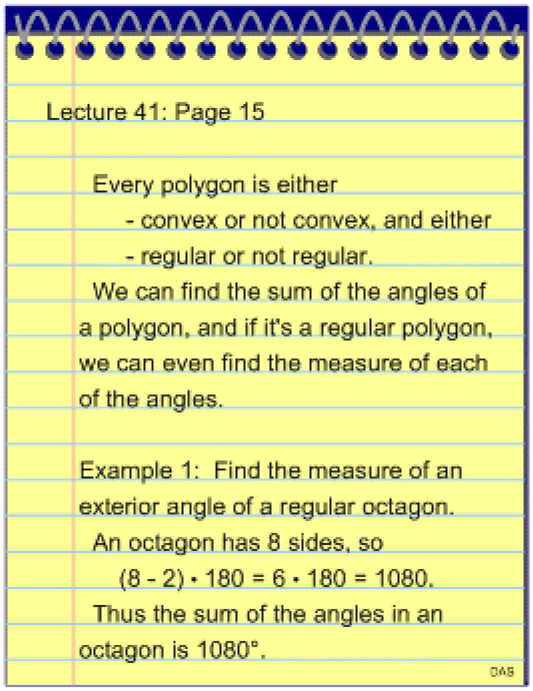
Lecture 41: Page 14

If we have an equilateral triangle, the three sides would have the same length. The three angles would also be the same; 180°/3 = 60°:

180° - 60° = 120°

The measure of an exterior angle of an equilateral triangle is 120°.

CEMP041-15



Lecture 41: Page 15

Every polygon is either
- convex or not convex, and either
- regular or not regular.

We can find the sum of the angles of a polygon, and if it's a regular polygon, we can even find the measure of each of the angles.

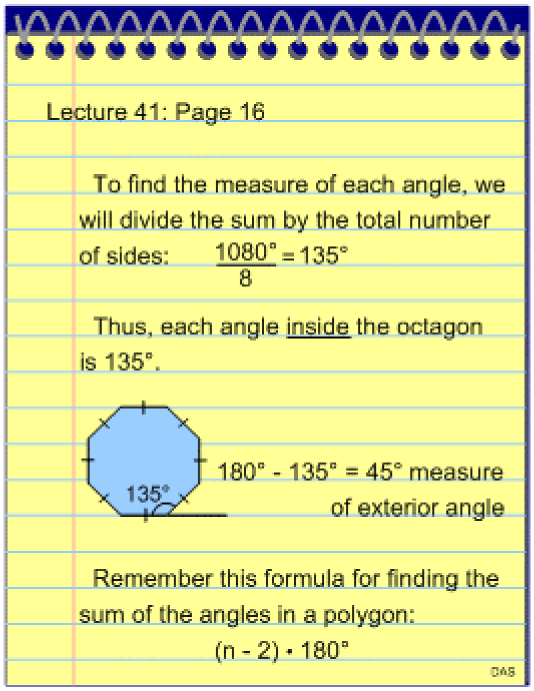
Example 1: Find the measure of an exterior angle of a regular octagon.

An octagon has 8 sides, so

$(8 - 2) \cdot 180 = 6 \cdot 180 = 1080.$

Thus the sum of the angles in an octagon is 1080°.

CEMP041-16



Lecture 41: Page 16

To find the measure of each angle, we will divide the sum by the total number of sides: $\frac{1080°}{8} = 135°$

Thus, each angle inside the octagon is 135°.

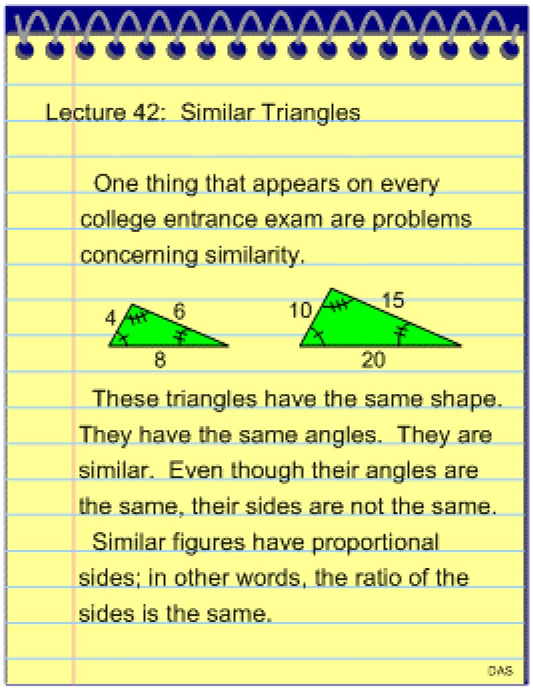
180° - 135° = 45° measure of exterior angle

Remember this formula for finding the sum of the angles in a polygon:

$(n - 2) \cdot 180°$

Lecture 42 Notes

CEMP042-01



Lecture 42: Similar Triangles

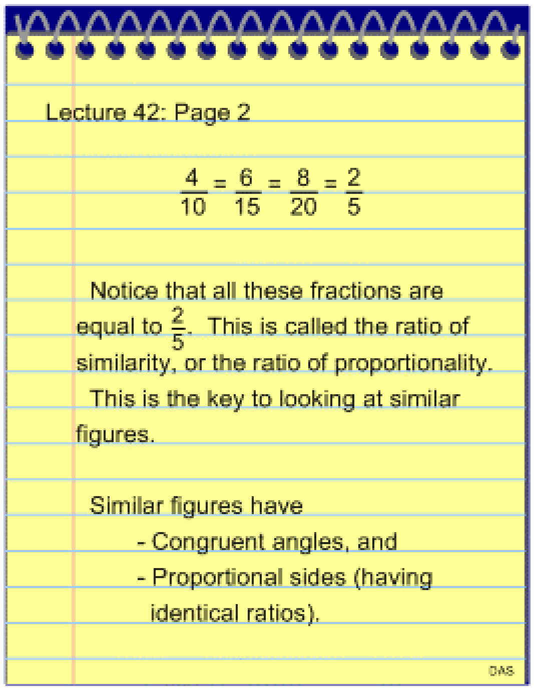
One thing that appears on every college entrance exam are problems concerning similarity.

These triangles have the same shape. They have the same angles. They are similar. Even though their angles are the same, their sides are not the same.

Similar figures have proportional sides; in other words, the ratio of the sides is the same.

DAS

CEMP042-02



Lecture 42: Page 2

$$\frac{4}{10} = \frac{6}{15} = \frac{8}{20} = \frac{2}{5}$$

Notice that all these fractions are equal to $\frac{2}{5}$. This is called the ratio of similarity, or the ratio of proportionality.

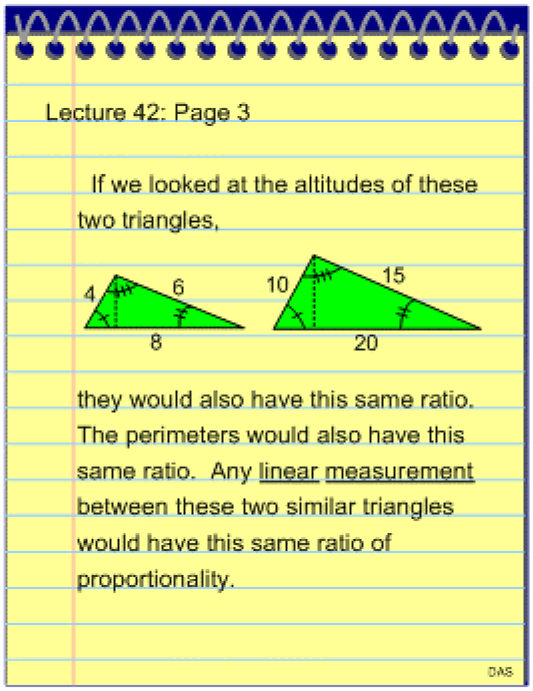
This is the key to looking at similar figures.

Similar figures have

- Congruent angles, and
- Proportional sides (having identical ratios).

DAS

CEMP042-03



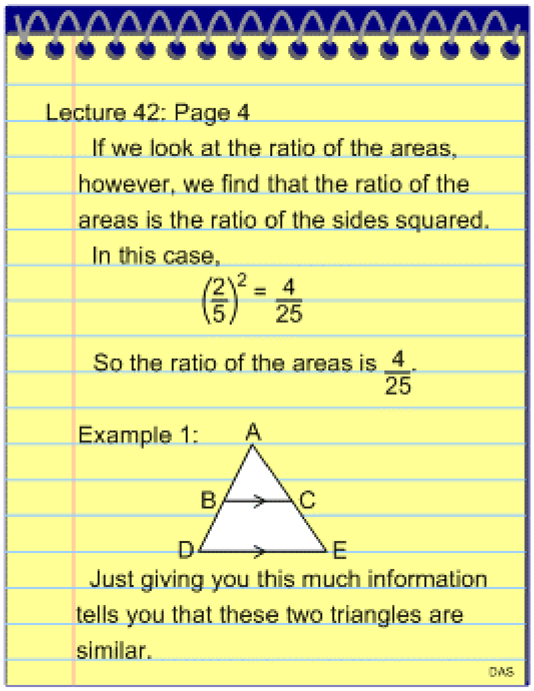
Lecture 42: Page 3

If we looked at the altitudes of these two triangles,

they would also have this same ratio. The perimeters would also have this same ratio. Any linear measurement between these two similar triangles would have this same ratio of proportionality.

DAS

CEMP042-04



Lecture 42: Page 4

If we look at the ratio of the areas, however, we find that the ratio of the areas is the ratio of the sides squared.

In this case,

$$\left(\frac{2}{5}\right)^2 = \frac{4}{25}$$

So the ratio of the areas is $\frac{4}{25}$.

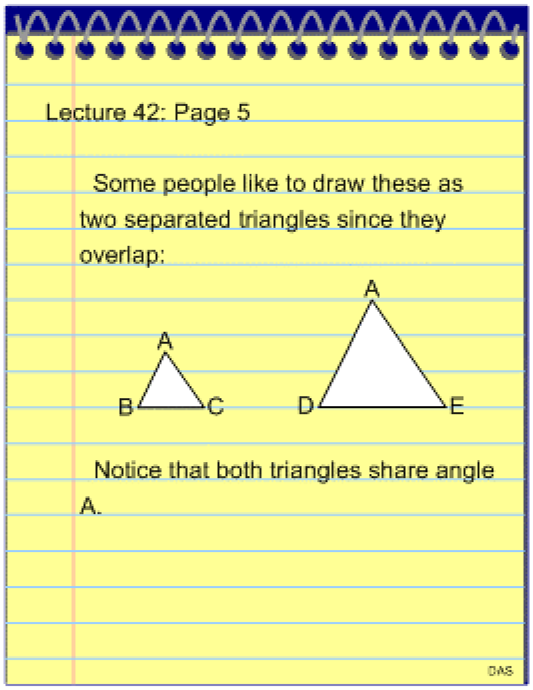
Example 1:

Just giving you this much information tells you that these two triangles are similar.

DAS

Lecture 42 Notes, Continued

CEMP042-05

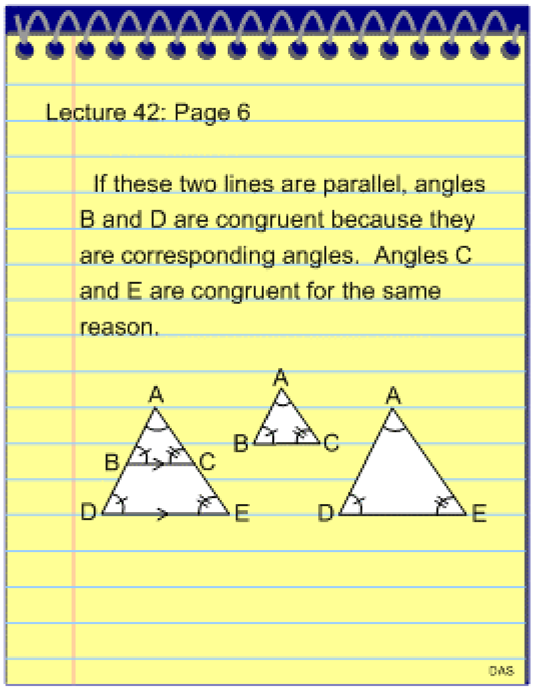


Lecture 42: Page 5

Some people like to draw these as two separated triangles since they overlap:

Notice that both triangles share angle A.

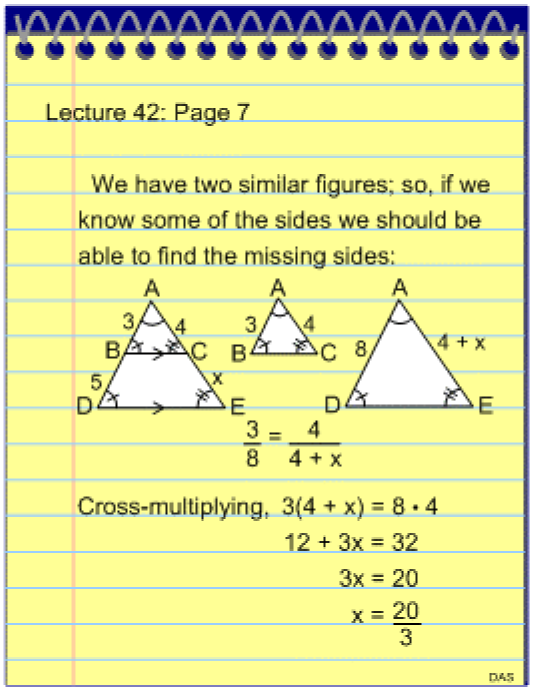
CEMP042-06



Lecture 42: Page 6

If these two lines are parallel, angles B and D are congruent because they are corresponding angles. Angles C and E are congruent for the same reason.

CEMP042-07



Lecture 42: Page 7

We have two similar figures; so, if we know some of the sides we should be able to find the missing sides:

$$\frac{3}{8} = \frac{4}{4 + x}$$

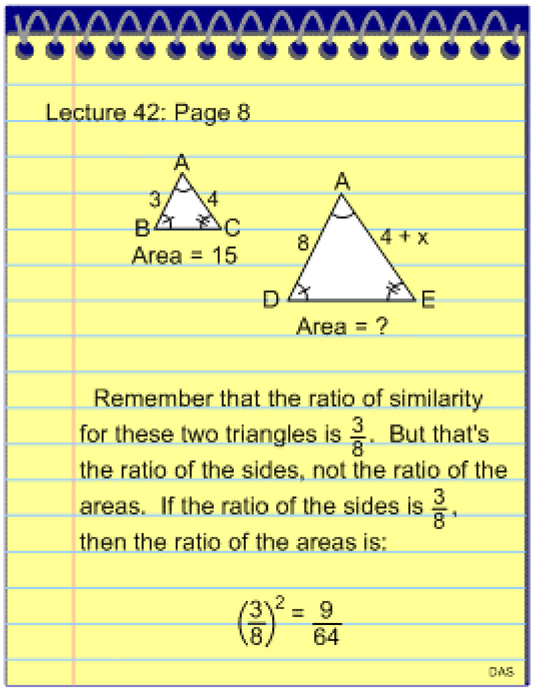
Cross-multiplying, $3(4 + x) = 8 \cdot 4$

$$12 + 3x = 32$$

$$3x = 20$$

$$x = \frac{20}{3}$$

CEMP042-08



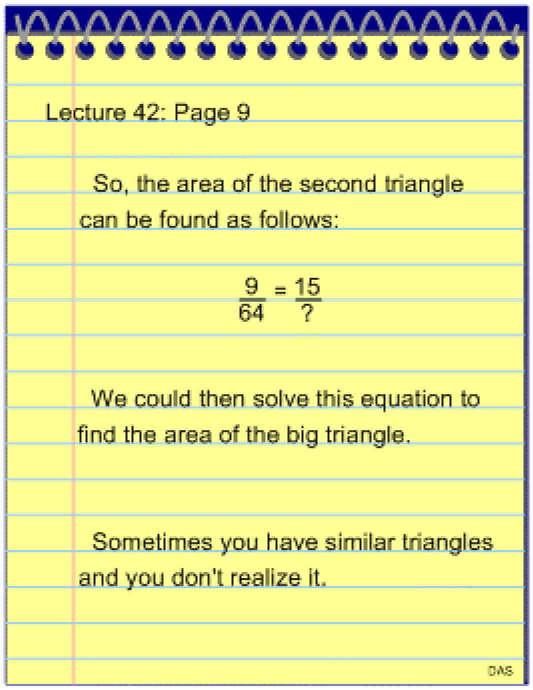
Lecture 42: Page 8

Remember that the ratio of similarity for these two triangles is $\frac{3}{8}$. But that's the ratio of the sides, not the ratio of the areas. If the ratio of the sides is $\frac{3}{8}$, then the ratio of the areas is:

$$\left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

Lecture 42 Notes, Continued

CEMP042-09



Lecture 42: Page 9

So, the area of the second triangle can be found as follows:

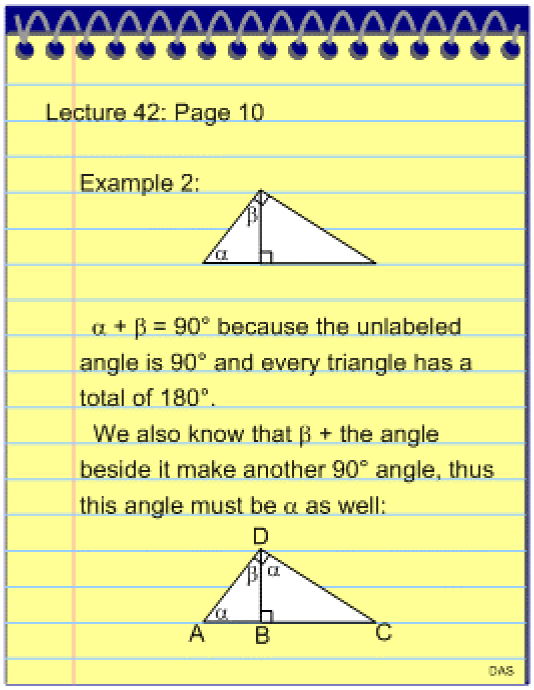
$$\frac{9}{64} = \frac{15}{?}$$

We could then solve this equation to find the area of the big triangle.

Sometimes you have similar triangles and you don't realize it.

DAS

CEMP042-10



Lecture 42: Page 10

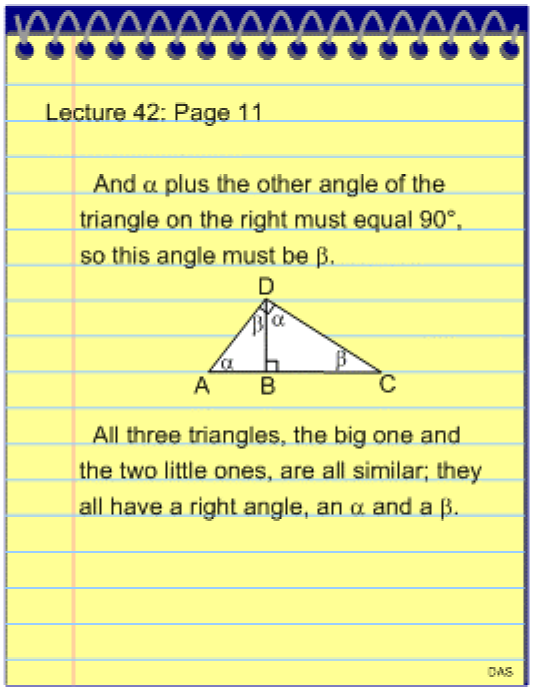
Example 2:

α + β = 90° because the unlabeled angle is 90° and every triangle has a total of 180°.

We also know that β + the angle beside it make another 90° angle, thus this angle must be α as well:

DAS

CEMP042-11



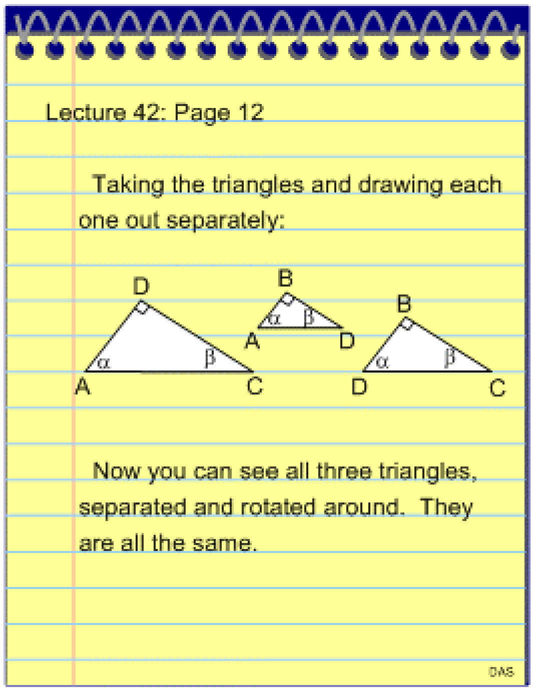
Lecture 42: Page 11

And α plus the other angle of the triangle on the right must equal 90°, so this angle must be β.

All three triangles, the big one and the two little ones, are all similar; they all have a right angle, an α and a β.

DAS

CEMP042-12



Lecture 42: Page 12

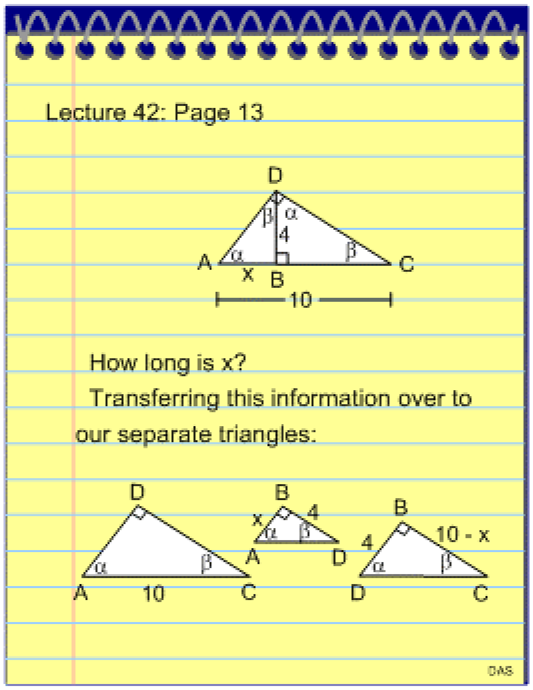
Taking the triangles and drawing each one out separately:

Now you can see all three triangles, separated and rotated around. They are all the same.

DAS

Lecture 42 Notes, Continued

CEMP042-13

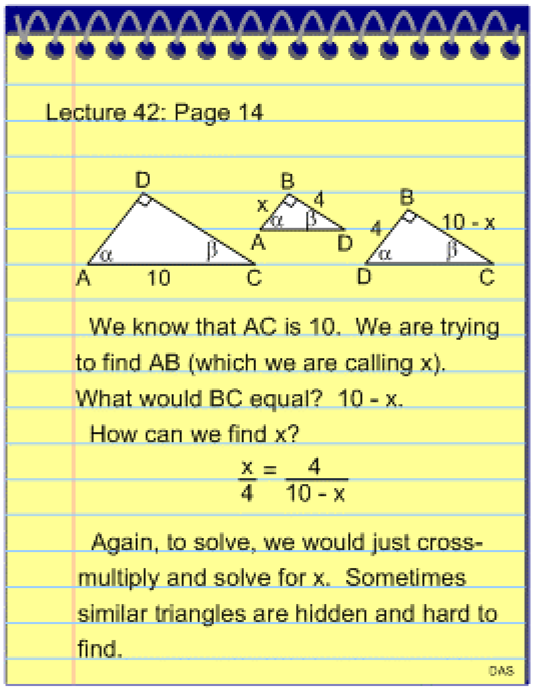


Lecture 42: Page 13

How long is x?

Transferring this information over to our separate triangles:

CEMP042-14



Lecture 42: Page 14

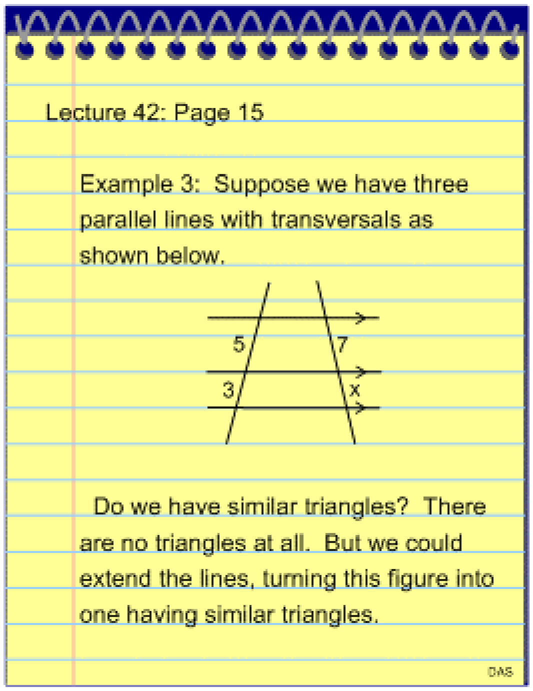
We know that AC is 10. We are trying to find AB (which we are calling x). What would BC equal? 10 - x.

How can we find x?

$$\frac{x}{4} = \frac{4}{10 - x}$$

Again, to solve, we would just cross-multiply and solve for x. Sometimes similar triangles are hidden and hard to find.

CEMP042-15

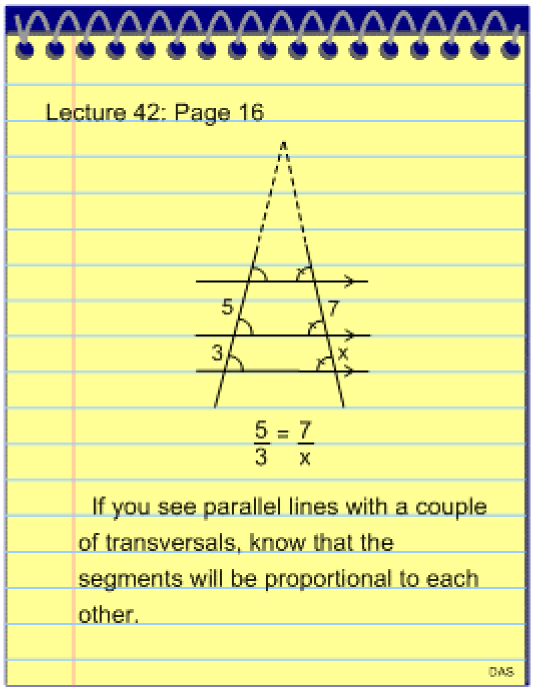


Lecture 42: Page 15

Example 3: Suppose we have three parallel lines with transversals as shown below.

Do we have similar triangles? There are no triangles at all. But we could extend the lines, turning this figure into one having similar triangles.

CEMP042-16



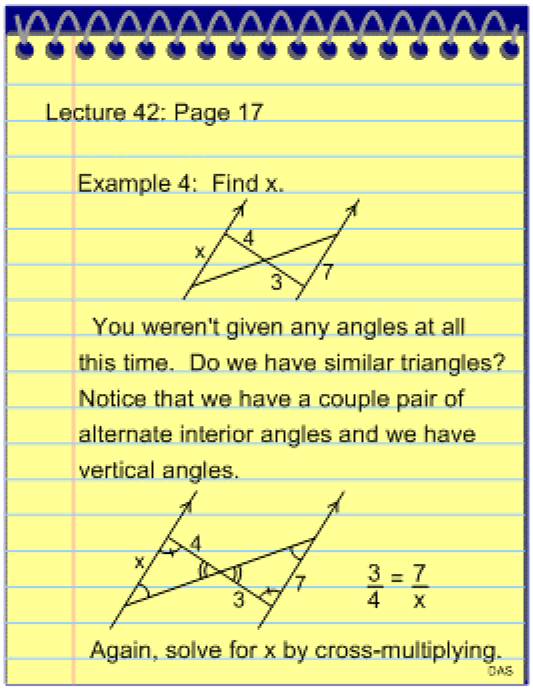
Lecture 42: Page 16

$$\frac{5}{3} = \frac{7}{x}$$

If you see parallel lines with a couple of transversals, know that the segments will be proportional to each other.

Lecture 42 Notes, Continued

CEMP042-17



Lecture 42: Page 17

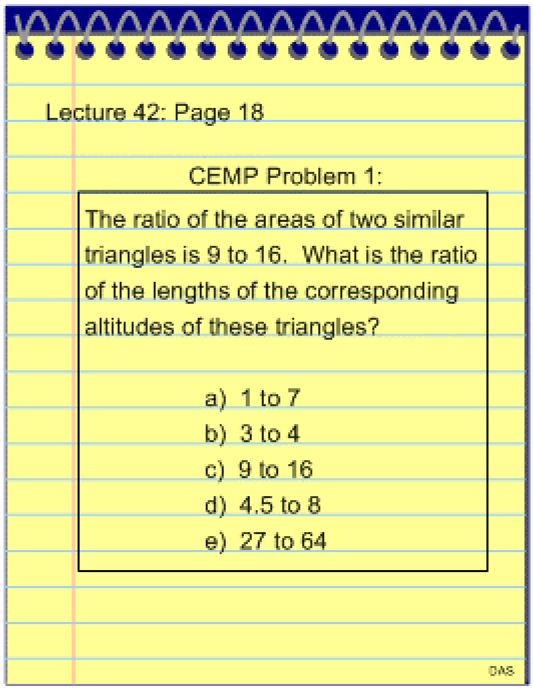
Example 4: Find x.

You weren't given any angles at all this time. Do we have similar triangles? Notice that we have a couple pair of alternate interior angles and we have vertical angles.

$\frac{3}{4} = \frac{7}{x}$

Again, solve for x by cross-multiplying.

CEMP042-18



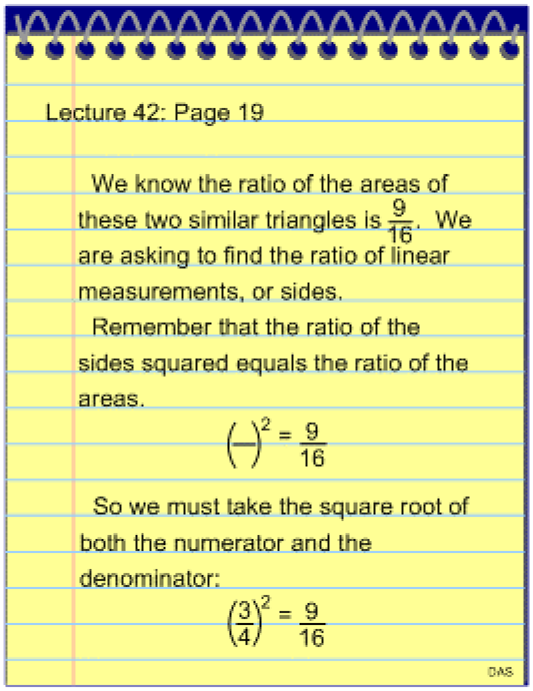
Lecture 42: Page 18

CEMP Problem 1:

The ratio of the areas of two similar triangles is 9 to 16. What is the ratio of the lengths of the corresponding altitudes of these triangles?

a) 1 to 7
b) 3 to 4
c) 9 to 16
d) 4.5 to 8
e) 27 to 64

CEMP042-19



Lecture 42: Page 19

We know the ratio of the areas of these two similar triangles is $\frac{9}{16}$. We are asking to find the ratio of linear measurements, or sides.

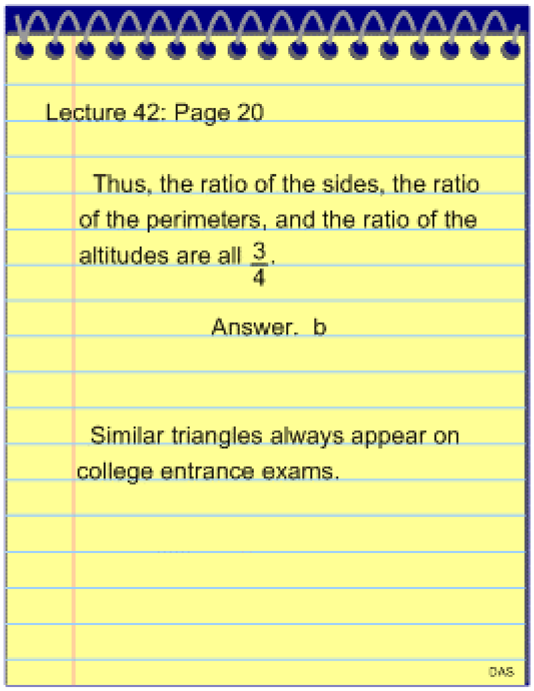
Remember that the ratio of the sides squared equals the ratio of the areas.

$$\left(\frac{}{}\right)^2 = \frac{9}{16}$$

So we must take the square root of both the numerator and the denominator:

$$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

CEMP042-20



Lecture 42: Page 20

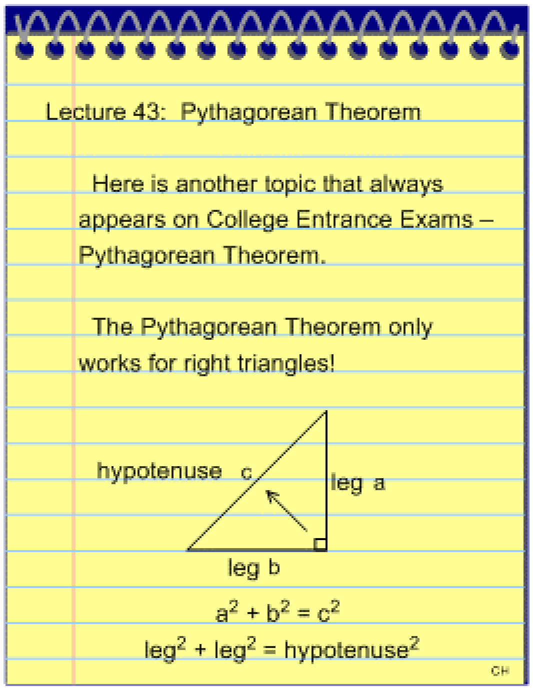
Thus, the ratio of the sides, the ratio of the perimeters, and the ratio of the altitudes are all $\frac{3}{4}$.

Answer. b

Similar triangles always appear on college entrance exams.

Lecture 43 Notes

CEMP043-01



Lecture 43: Pythagorean Theorem

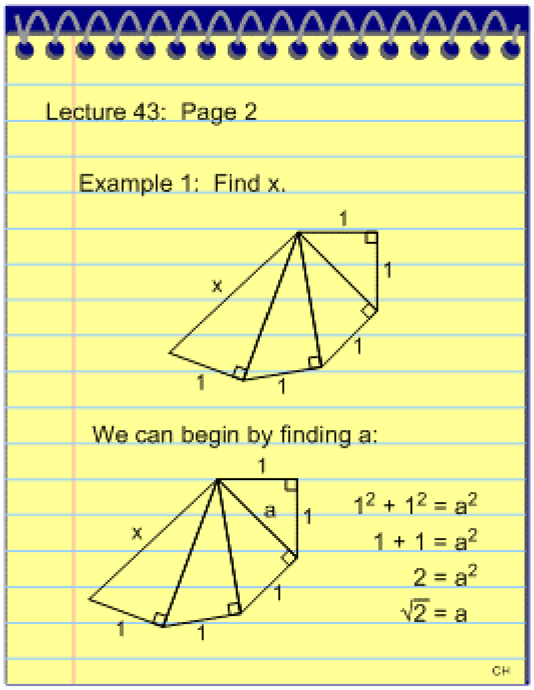
Here is another topic that always appears on College Entrance Exams – Pythagorean Theorem.

The Pythagorean Theorem only works for right triangles!

$a^2 + b^2 = c^2$

$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$

CEMP043-02



Lecture 43: Page 2

Example 1: Find x.

We can begin by finding a:

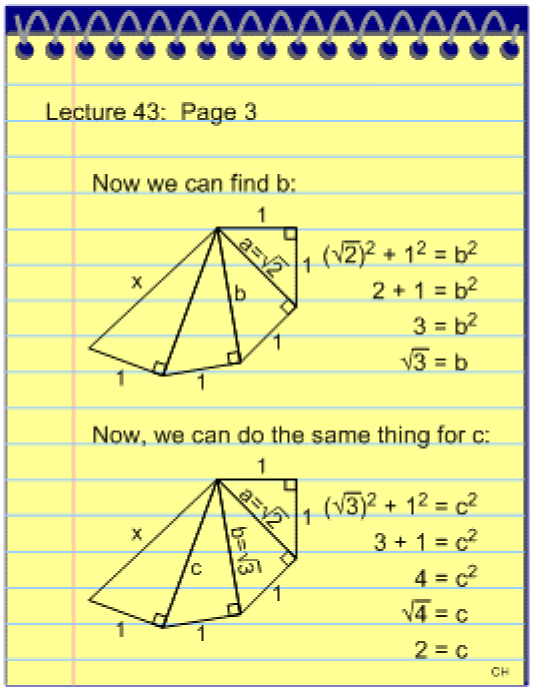
$1^2 + 1^2 = a^2$

$1 + 1 = a^2$

$2 = a^2$

$\sqrt{2} = a$

CEMP043-03



Lecture 43: Page 3

Now we can find b:

$(\sqrt{2})^2 + 1^2 = b^2$

$2 + 1 = b^2$

$3 = b^2$

$\sqrt{3} = b$

Now, we can do the same thing for c:

$(\sqrt{3})^2 + 1^2 = c^2$

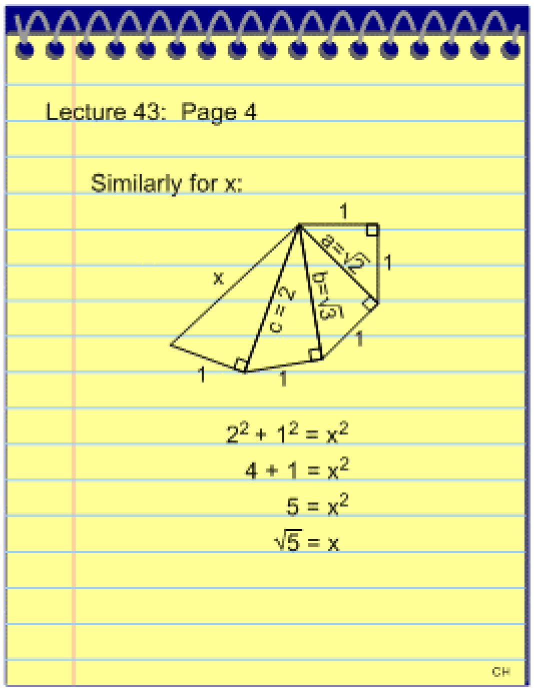
$3 + 1 = c^2$

$4 = c^2$

$\sqrt{4} = c$

$2 = c$

CEMP043-04



Lecture 43: Page 4

Similarly for x:

$2^2 + 1^2 = x^2$

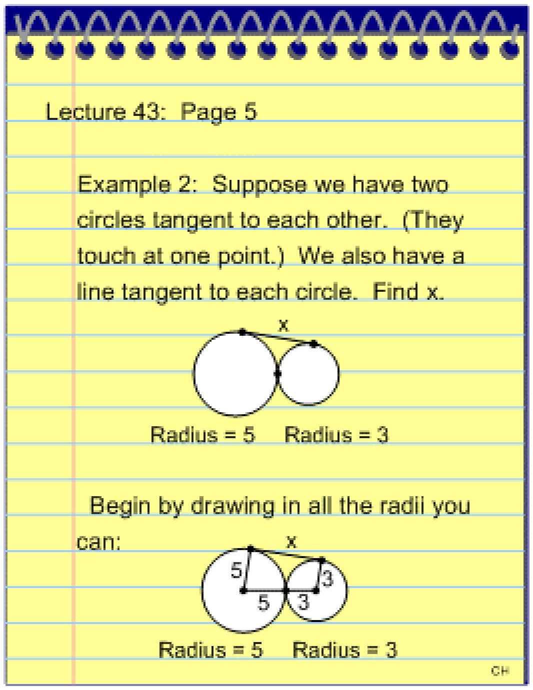
$4 + 1 = x^2$

$5 = x^2$

$\sqrt{5} = x$

Lecture 43 Notes, Continued

CEMP043-05



Lecture 43: Page 5

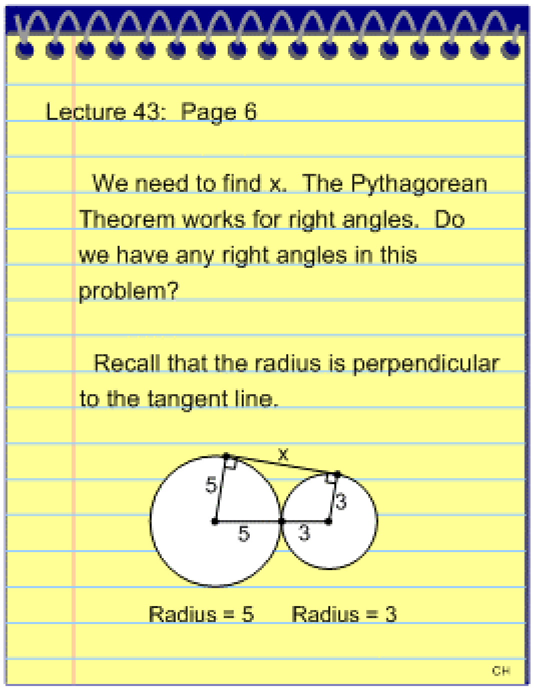
Example 2: Suppose we have two circles tangent to each other. (They touch at one point.) We also have a line tangent to each circle. Find x.

Radius = 5 Radius = 3

Begin by drawing in all the radii you can:

Radius = 5 Radius = 3

CEMP043-06



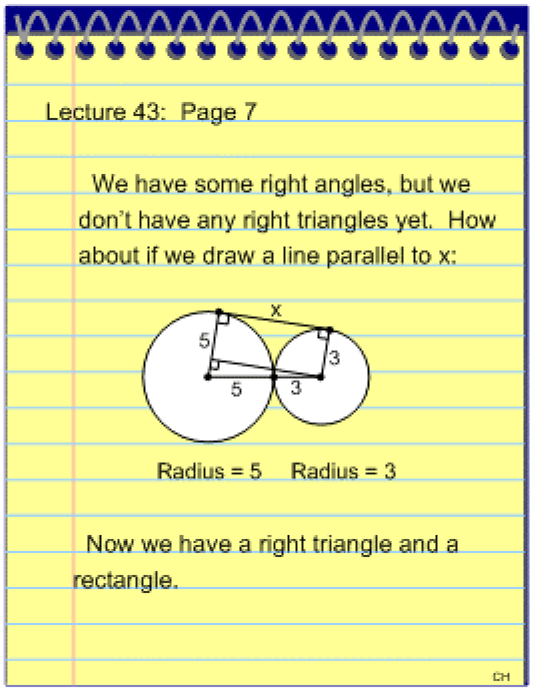
Lecture 43: Page 6

We need to find x. The Pythagorean Theorem works for right angles. Do we have any right angles in this problem?

Recall that the radius is perpendicular to the tangent line.

Radius = 5 Radius = 3

CEMP043-07



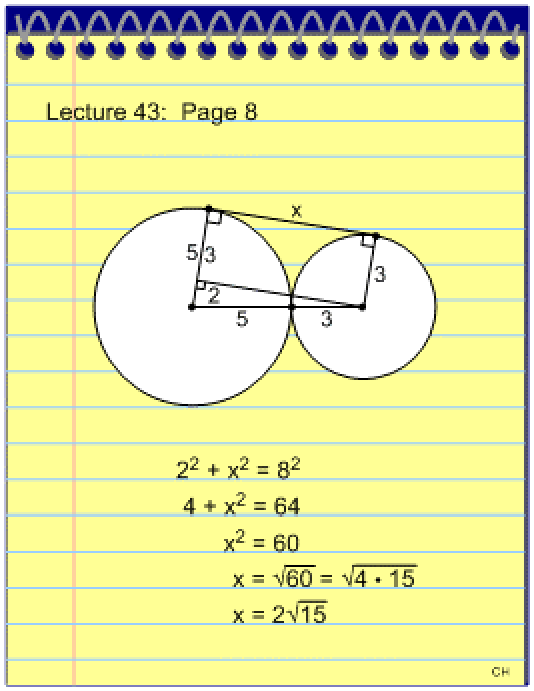
Lecture 43: Page 7

We have some right angles, but we don't have any right triangles yet. How about if we draw a line parallel to x:

Radius = 5 Radius = 3

Now we have a right triangle and a rectangle.

CEMP043-08



Lecture 43: Page 8

$2^2 + x^2 = 8^2$

$4 + x^2 = 64$

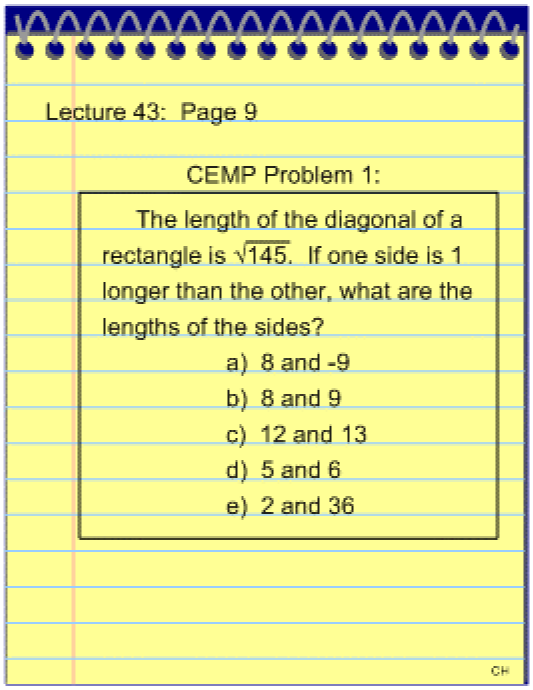
$x^2 = 60$

$x = \sqrt{60} = \sqrt{4 \cdot 15}$

$x = 2\sqrt{15}$

Lecture 43 Notes, Continued

CEMP043-09



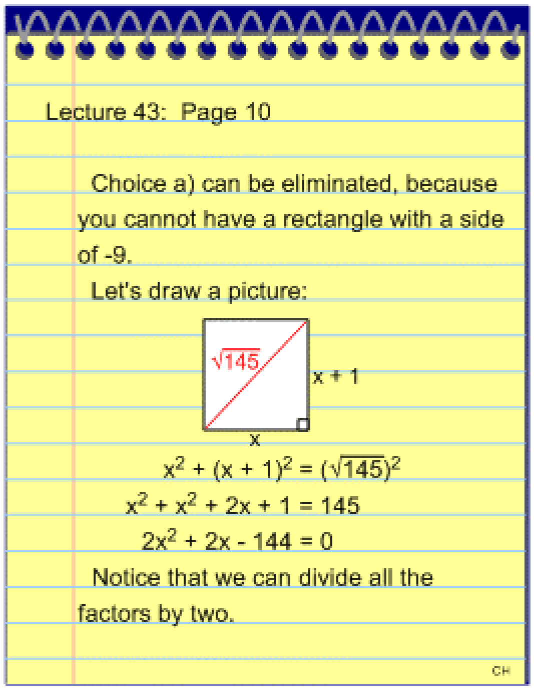
Lecture 43: Page 9

CEMP Problem 1:

The length of the diagonal of a rectangle is $\sqrt{145}$. If one side is 1 longer than the other, what are the lengths of the sides?

a) 8 and -9
b) 8 and 9
c) 12 and 13
d) 5 and 6
e) 2 and 36

CEMP043-10



Lecture 43: Page 10

Choice a) can be eliminated, because you cannot have a rectangle with a side of -9.

Let's draw a picture:

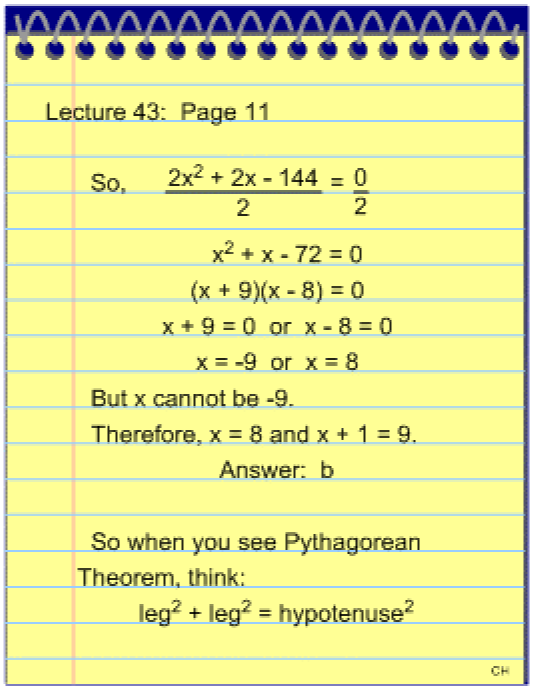
$$x^2 + (x + 1)^2 = (\sqrt{145})^2$$

$$x^2 + x^2 + 2x + 1 = 145$$

$$2x^2 + 2x - 144 = 0$$

Notice that we can divide all the factors by two.

CEMP043-11



Lecture 43: Page 11

So, $$\frac{2x^2 + 2x - 144}{2} = \frac{0}{2}$$

$$x^2 + x - 72 = 0$$

$$(x + 9)(x - 8) = 0$$

$$x + 9 = 0 \text{ or } x - 8 = 0$$

$$x = -9 \text{ or } x = 8$$

But x cannot be -9.

Therefore, $x = 8$ and $x + 1 = 9$.

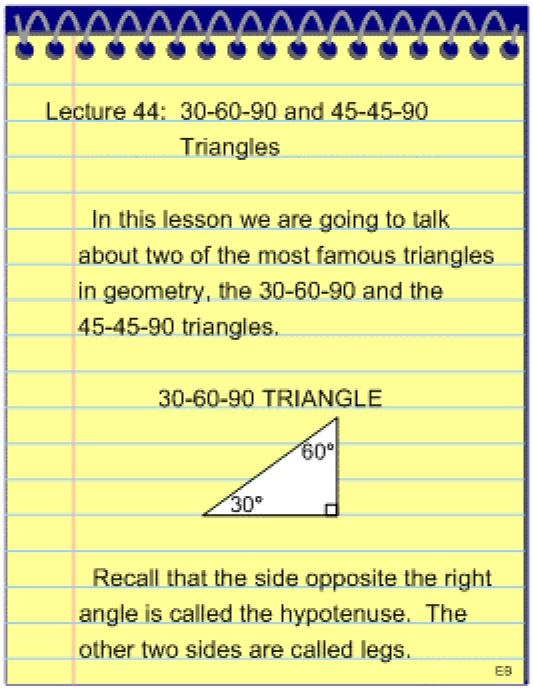
Answer: b

So when you see Pythagorean Theorem, think:

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

Lecture 44 Notes

CEMP044-01



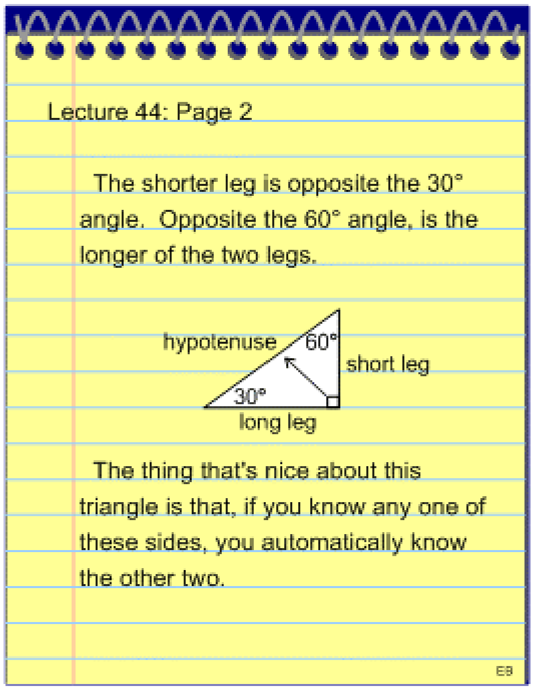
Lecture 44: 30-60-90 and 45-45-90 Triangles

In this lesson we are going to talk about two of the most famous triangles in geometry, the 30-60-90 and the 45-45-90 triangles.

30-60-90 TRIANGLE

Recall that the side opposite the right angle is called the hypotenuse. The other two sides are called legs.

CEMP044-02

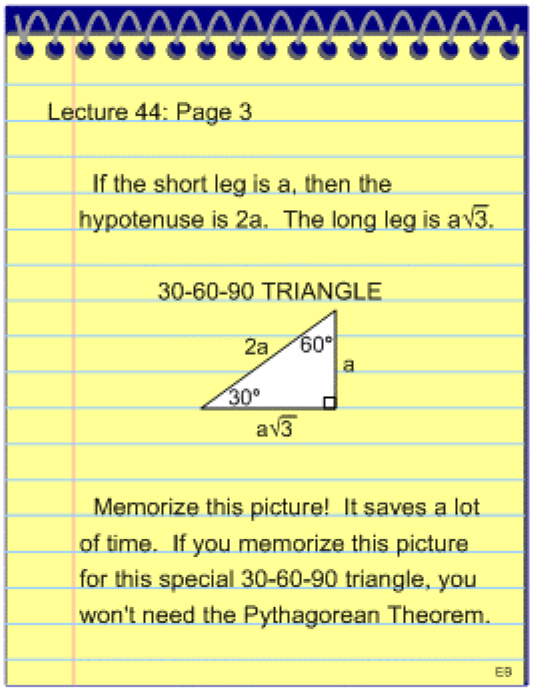


Lecture 44: Page 2

The shorter leg is opposite the 30° angle. Opposite the 60° angle, is the longer of the two legs.

The thing that's nice about this triangle is that, if you know any one of these sides, you automatically know the other two.

CEMP044-03



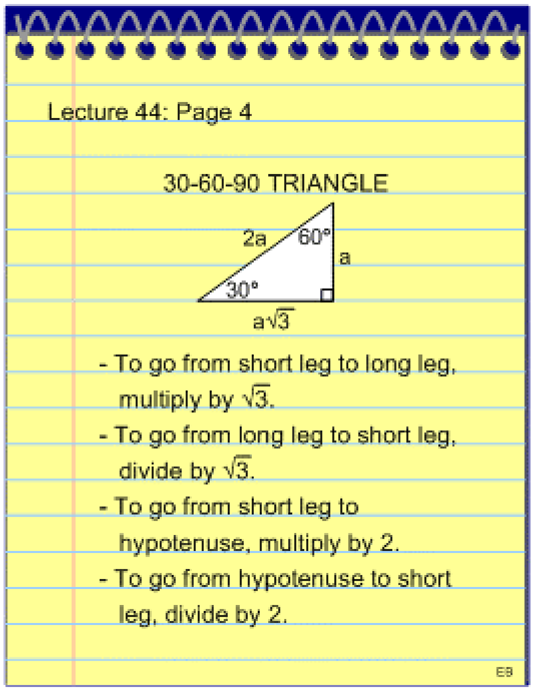
Lecture 44: Page 3

If the short leg is a, then the hypotenuse is 2a. The long leg is a√3.

30-60-90 TRIANGLE

Memorize this picture! It saves a lot of time. If you memorize this picture for this special 30-60-90 triangle, you won't need the Pythagorean Theorem.

CEMP044-04



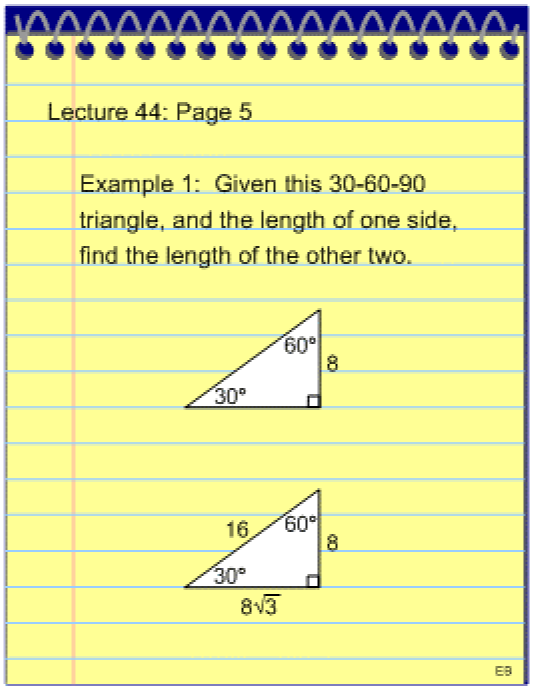
Lecture 44: Page 4

30-60-90 TRIANGLE

- To go from short leg to long leg, multiply by √3.
- To go from long leg to short leg, divide by √3.
- To go from short leg to hypotenuse, multiply by 2.
- To go from hypotenuse to short leg, divide by 2.

Lecture 44 Notes, Continued

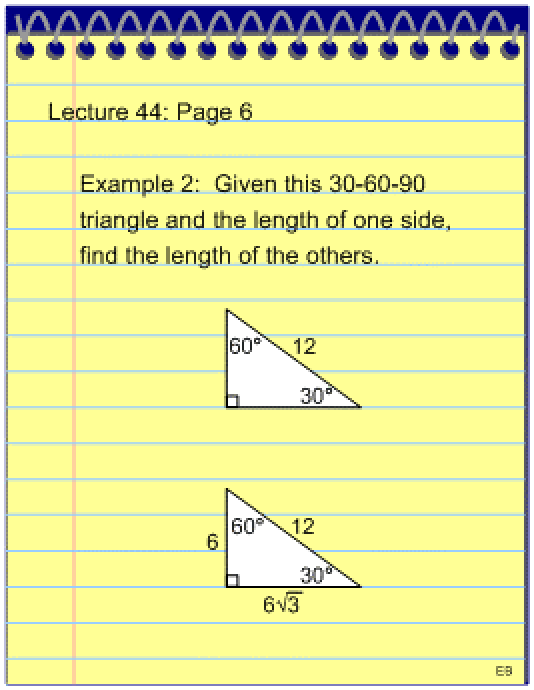
CEMP044-05



Lecture 44: Page 5

Example 1: Given this 30-60-90 triangle, and the length of one side, find the length of the other two.

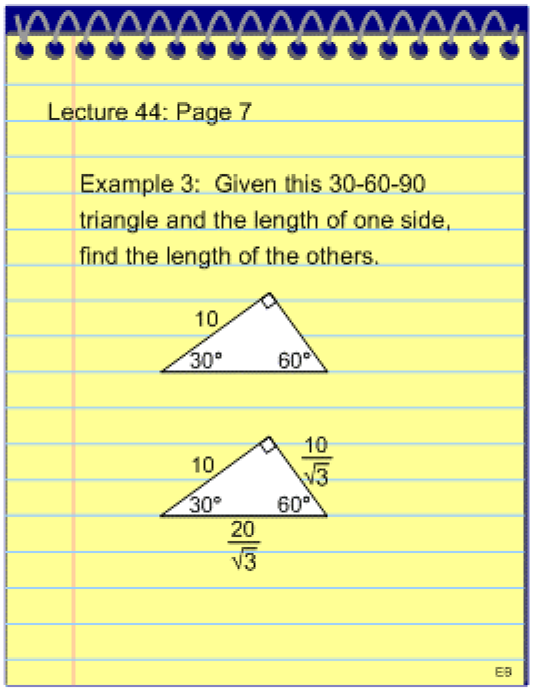
CEMP044-06



Lecture 44: Page 6

Example 2: Given this 30-60-90 triangle and the length of one side, find the length of the others.

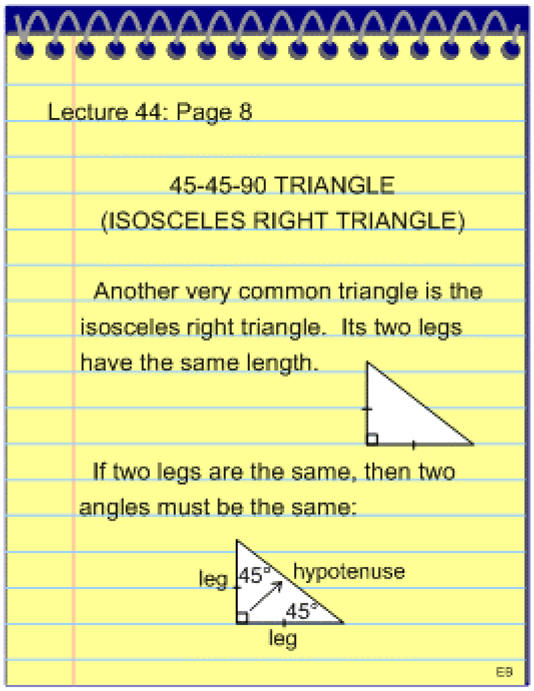
CEMP044-07



Lecture 44: Page 7

Example 3: Given this 30-60-90 triangle and the length of one side, find the length of the others.

CEMP044-08



Lecture 44: Page 8

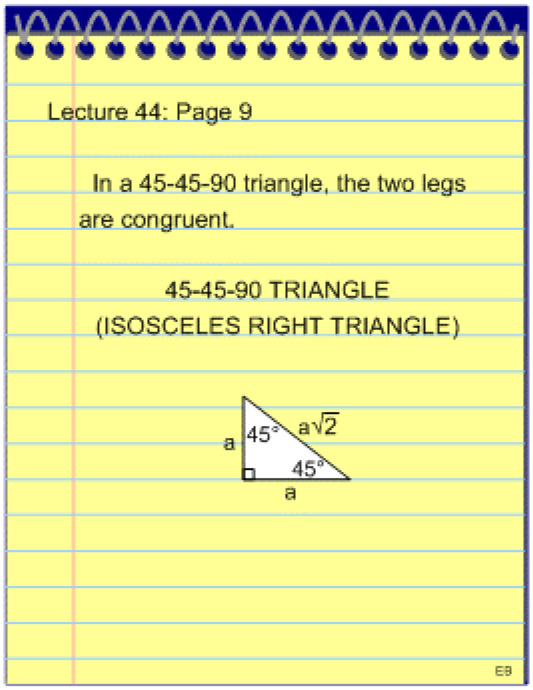
45-45-90 TRIANGLE
(ISOSCELES RIGHT TRIANGLE)

Another very common triangle is the isosceles right triangle. Its two legs have the same length.

If two legs are the same, then two angles must be the same:

Lecture 44 Notes, Continued

CEMP044-09

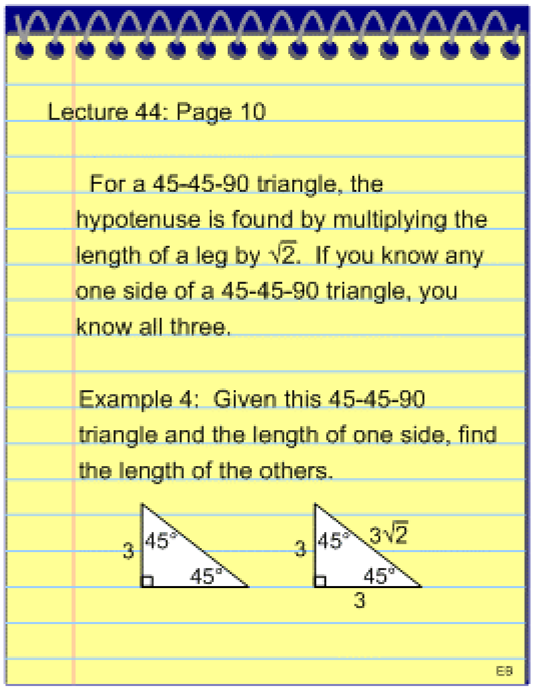


Lecture 44: Page 9

In a 45-45-90 triangle, the two legs are congruent.

45-45-90 TRIANGLE
(ISOSCELES RIGHT TRIANGLE)

CEMP044-10

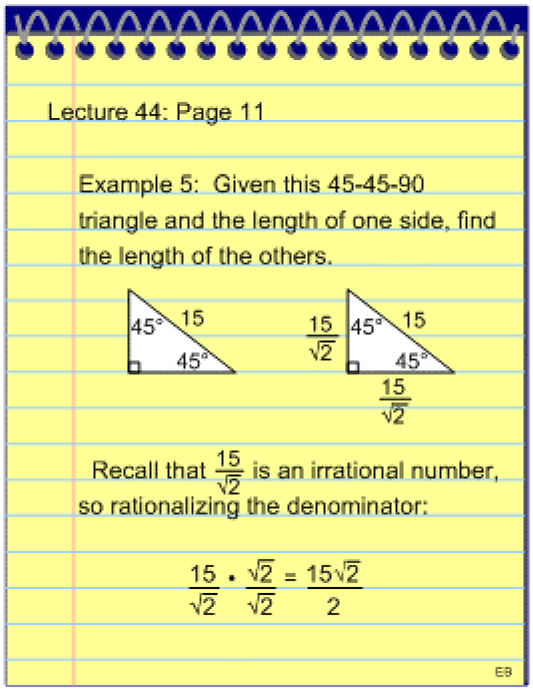


Lecture 44: Page 10

For a 45-45-90 triangle, the hypotenuse is found by multiplying the length of a leg by $\sqrt{2}$. If you know any one side of a 45-45-90 triangle, you know all three.

Example 4: Given this 45-45-90 triangle and the length of one side, find the length of the others.

CEMP044-11



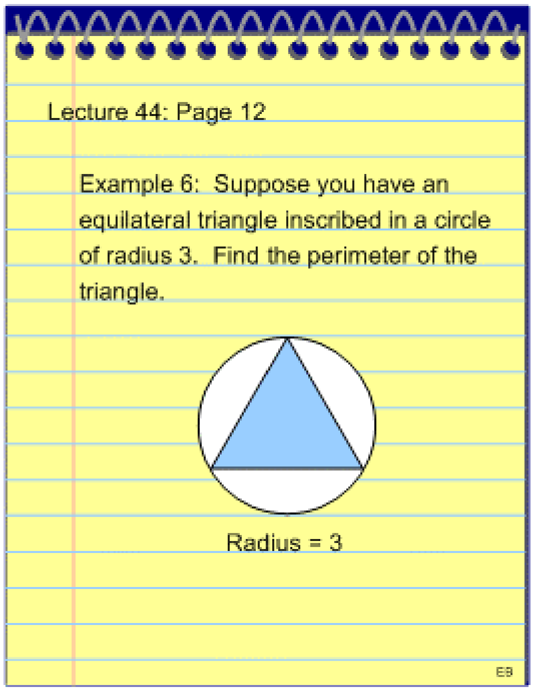
Lecture 44: Page 11

Example 5: Given this 45-45-90 triangle and the length of one side, find the length of the others.

Recall that $\frac{15}{\sqrt{2}}$ is an irrational number, so rationalizing the denominator:

$$\frac{15}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{15\sqrt{2}}{2}$$

CEMP044-12



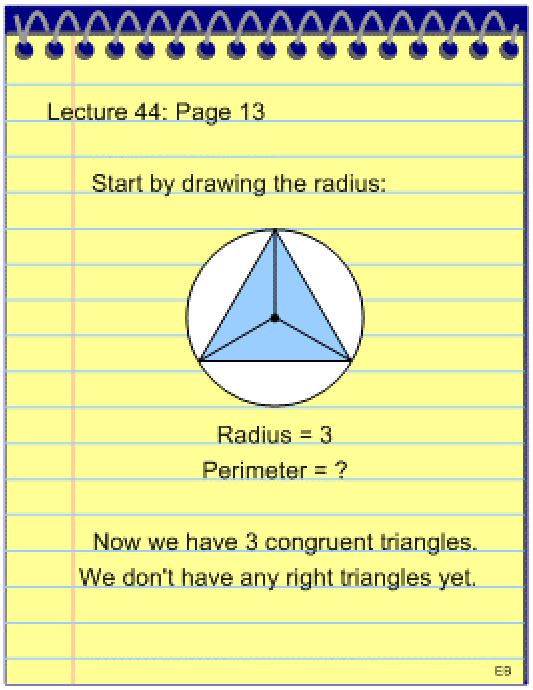
Lecture 44: Page 12

Example 6: Suppose you have an equilateral triangle inscribed in a circle of radius 3. Find the perimeter of the triangle.

Radius = 3

Lecture 44 Notes, Continued

CEMP044-13



Lecture 44: Page 13

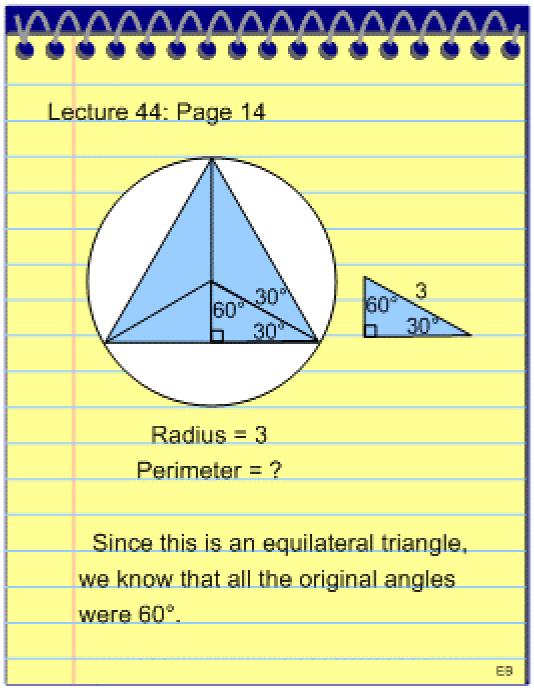
Start by drawing the radius:

Radius = 3

Perimeter = ?

Now we have 3 congruent triangles.
We don't have any right triangles yet.

CEMP044-14



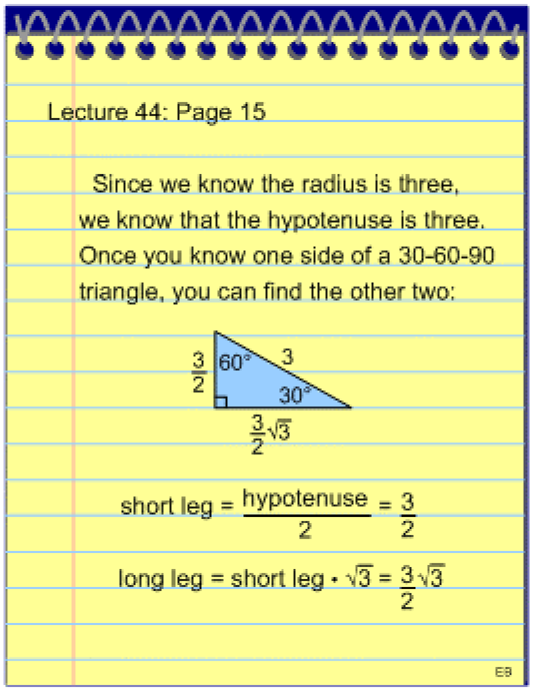
Lecture 44: Page 14

Radius = 3

Perimeter = ?

Since this is an equilateral triangle, we know that all the original angles were 60°.

CEMP044-15



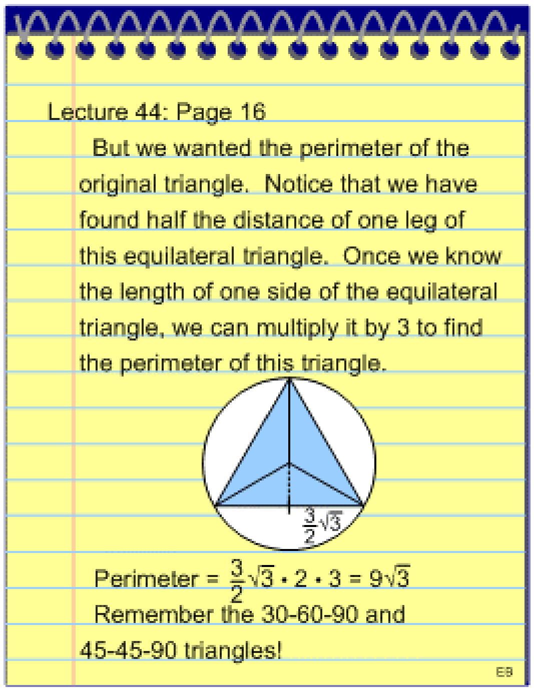
Lecture 44: Page 15

Since we know the radius is three, we know that the hypotenuse is three. Once you know one side of a 30-60-90 triangle, you can find the other two:

$$\text{short leg} = \frac{\text{hypotenuse}}{2} = \frac{3}{2}$$

$$\text{long leg} = \text{short leg} \cdot \sqrt{3} = \frac{3}{2}\sqrt{3}$$

CEMP044-16



Lecture 44: Page 16

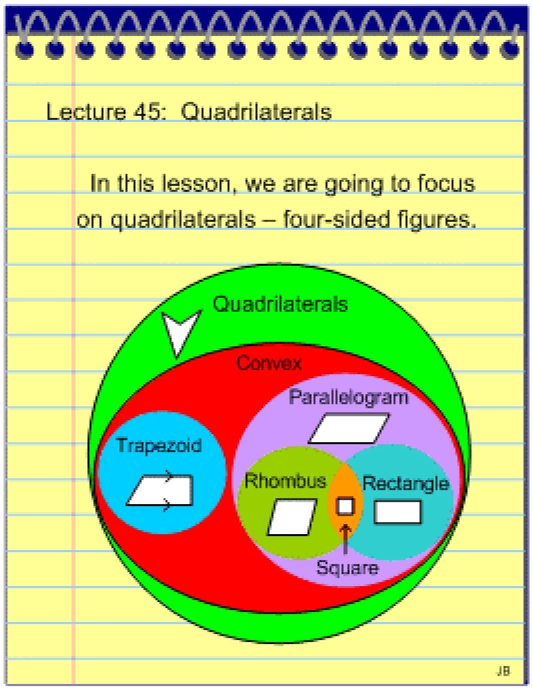
But we wanted the perimeter of the original triangle. Notice that we have found half the distance of one leg of this equilateral triangle. Once we know the length of one side of the equilateral triangle, we can multiply it by 3 to find the perimeter of this triangle.

$$\text{Perimeter} = \frac{3}{2}\sqrt{3} \cdot 2 \cdot 3 = 9\sqrt{3}$$

Remember the 30-60-90 and 45-45-90 triangles!

Lecture 45 Notes

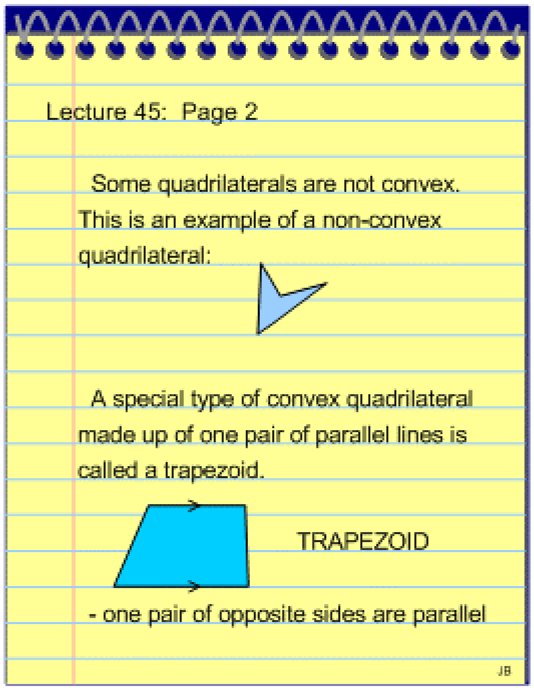
CEMP045-01



Lecture 45: Quadrilaterals

In this lesson, we are going to focus on quadrilaterals – four-sided figures.

CEMP045-02



Lecture 45: Page 2

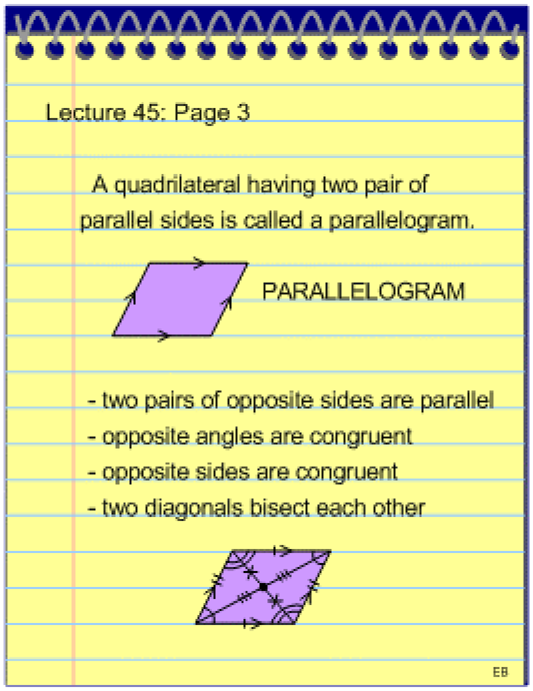
Some quadrilaterals are not convex. This is an example of a non-convex quadrilateral:

A special type of convex quadrilateral made up of one pair of parallel lines is called a trapezoid.

TRAPEZOID

- one pair of opposite sides are parallel

CEMP045-03



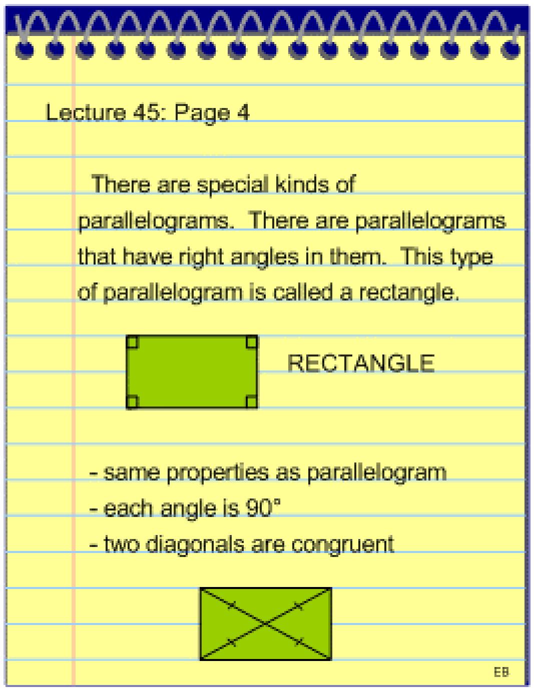
Lecture 45: Page 3

A quadrilateral having two pair of parallel sides is called a parallelogram.

PARALLELOGRAM

- two pairs of opposite sides are parallel
- opposite angles are congruent
- opposite sides are congruent
- two diagonals bisect each other

CEMP045-04



Lecture 45: Page 4

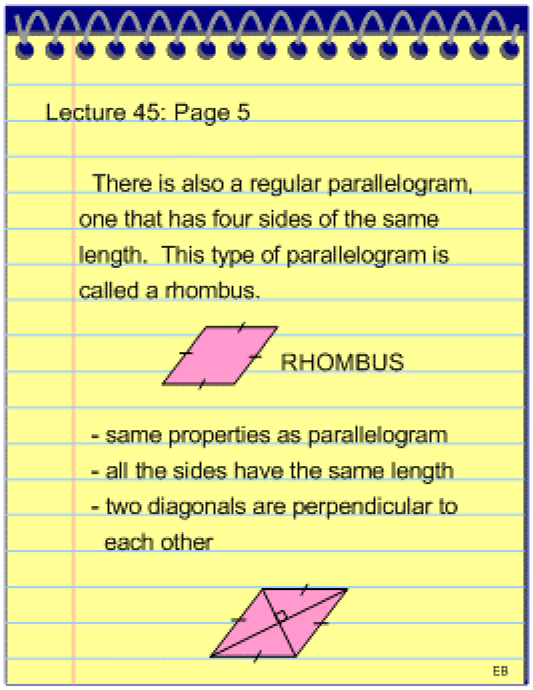
There are special kinds of parallelograms. There are parallelograms that have right angles in them. This type of parallelogram is called a rectangle.

RECTANGLE

- same properties as parallelogram
- each angle is 90°
- two diagonals are congruent

Lecture 45 Notes, Continued

CEMP045-05



Lecture 45: Page 5

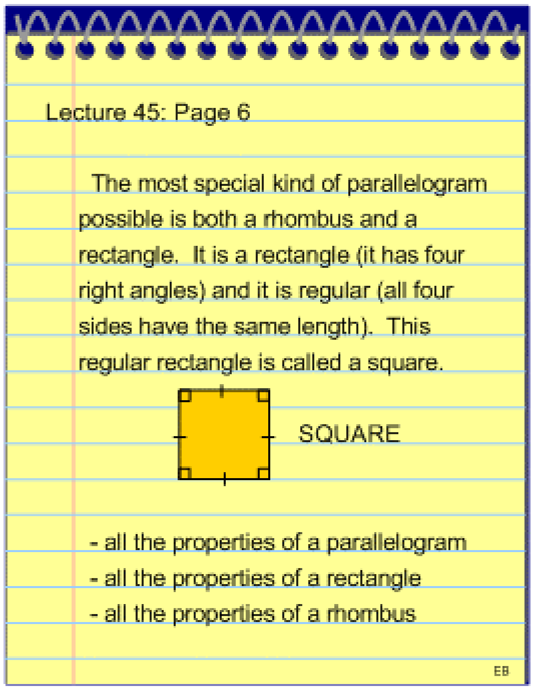
There is also a regular parallelogram, one that has four sides of the same length. This type of parallelogram is called a rhombus.

RHOMBUS

- same properties as parallelogram
- all the sides have the same length
- two diagonals are perpendicular to each other

EB

CEMP045-06



Lecture 45: Page 6

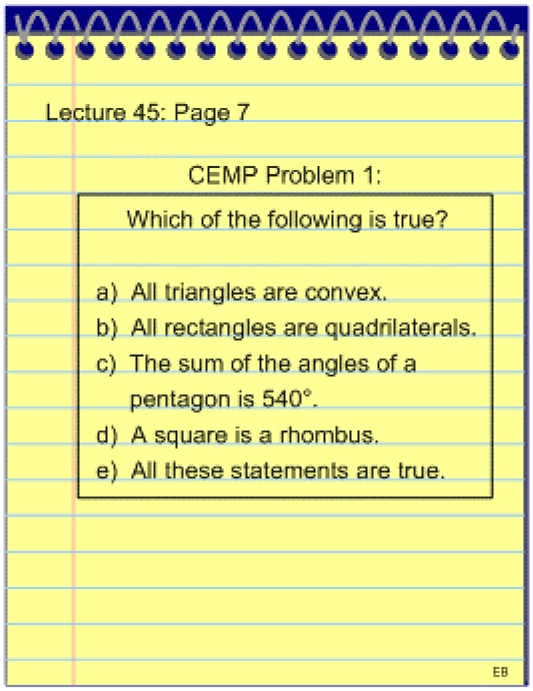
The most special kind of parallelogram possible is both a rhombus and a rectangle. It is a rectangle (it has four right angles) and it is regular (all four sides have the same length). This regular rectangle is called a square.

SQUARE

- all the properties of a parallelogram
- all the properties of a rectangle
- all the properties of a rhombus

EB

CEMP045-07



Lecture 45: Page 7

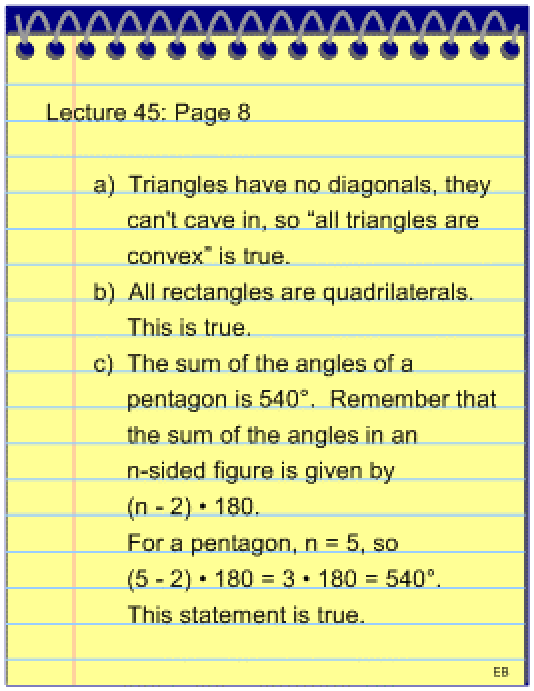
CEMP Problem 1:

Which of the following is true?

a) All triangles are convex.
b) All rectangles are quadrilaterals.
c) The sum of the angles of a pentagon is 540°.
d) A square is a rhombus.
e) All these statements are true.

EB

CEMP045-08



Lecture 45: Page 8

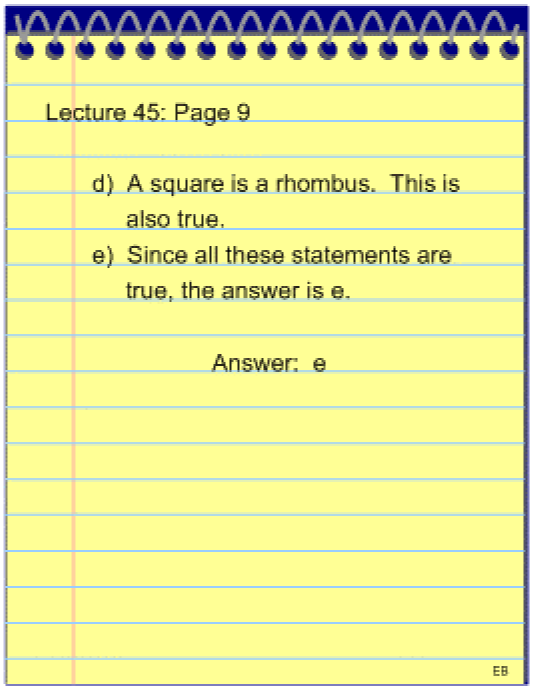
a) Triangles have no diagonals, they can't cave in, so "all triangles are convex" is true.

b) All rectangles are quadrilaterals. This is true.

c) The sum of the angles of a pentagon is 540°. Remember that the sum of the angles in an n-sided figure is given by $(n - 2) \cdot 180$.
For a pentagon, $n = 5$, so
$(5 - 2) \cdot 180 = 3 \cdot 180 = 540°$.
This statement is true.

EB

CEMP045-09



Lecture 45: Page 9

d) A square is a rhombus. This is also true.

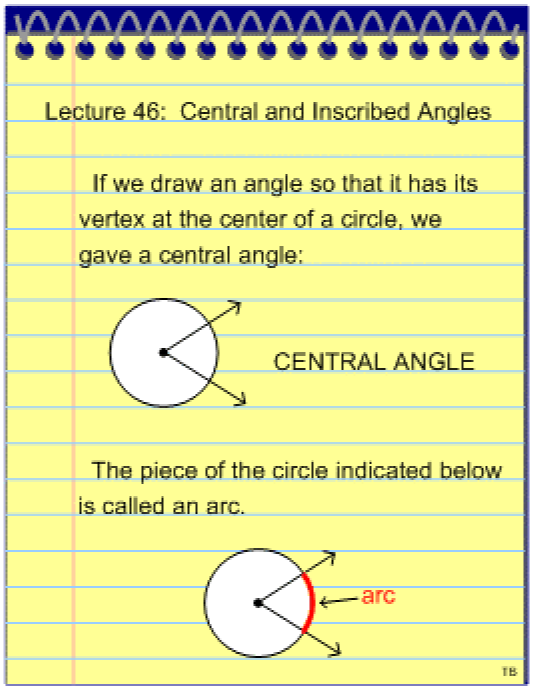
e) Since all these statements are true, the answer is e.

Answer: e

EB

Lecture 46 Notes

CEMP046-01



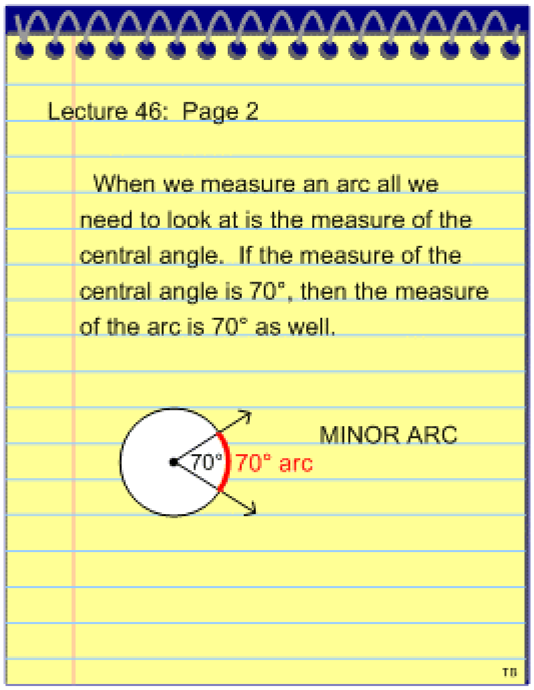
Lecture 46: Central and Inscribed Angles

If we draw an angle so that it has its vertex at the center of a circle, we gave a central angle:

CENTRAL ANGLE

The piece of the circle indicated below is called an arc.

CEMP046-02

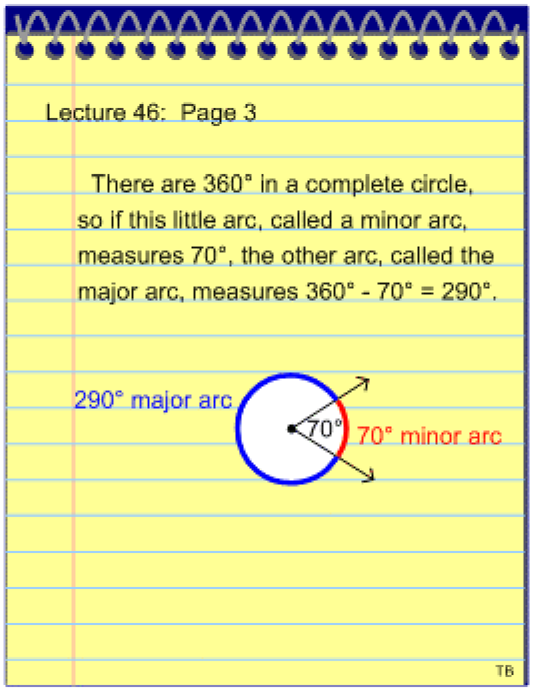


Lecture 46: Page 2

When we measure an arc all we need to look at is the measure of the central angle. If the measure of the central angle is 70°, then the measure of the arc is 70° as well.

MINOR ARC

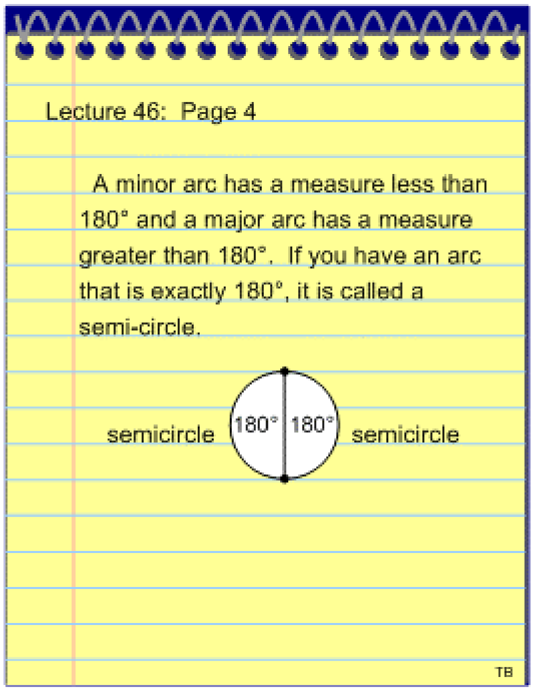
CEMP046-03



Lecture 46: Page 3

There are 360° in a complete circle, so if this little arc, called a minor arc, measures 70°, the other arc, called the major arc, measures 360° - 70° = 290°.

CEMP046-04

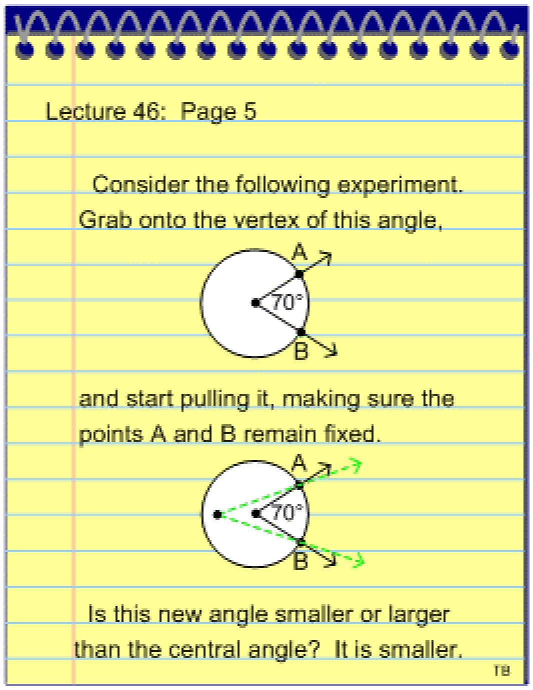


Lecture 46: Page 4

A minor arc has a measure less than 180° and a major arc has a measure greater than 180°. If you have an arc that is exactly 180°, it is called a semi-circle.

Lecture 46 Notes, Continued

CEMP046-05



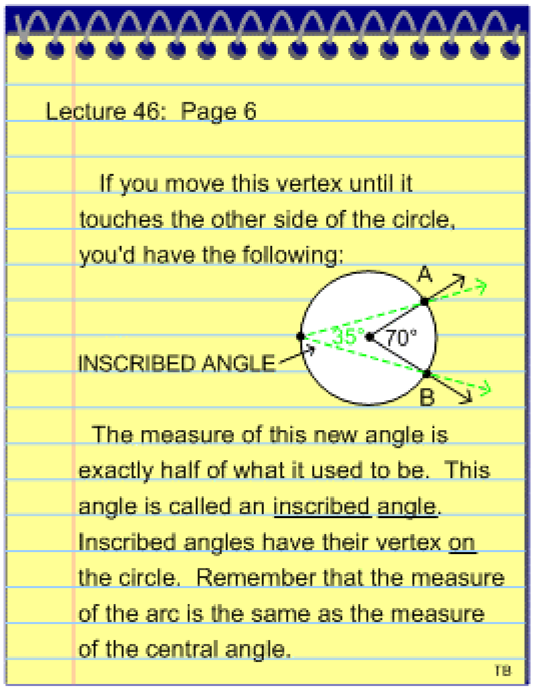
Lecture 46: Page 5

Consider the following experiment. Grab onto the vertex of this angle,

and start pulling it, making sure the points A and B remain fixed.

Is this new angle smaller or larger than the central angle? It is smaller.

CEMP046-06

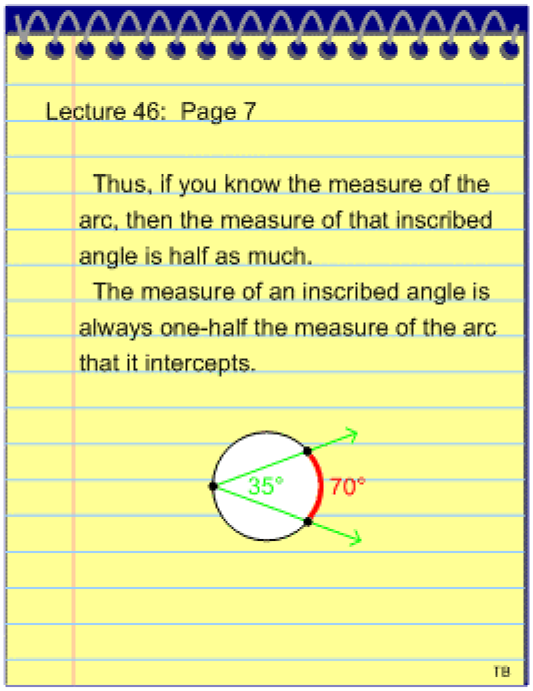


Lecture 46: Page 6

If you move this vertex until it touches the other side of the circle, you'd have the following:

The measure of this new angle is exactly half of what it used to be. This angle is called an <u>inscribed angle</u>. Inscribed angles have their vertex <u>on</u> the circle. Remember that the measure of the arc is the same as the measure of the central angle.

CEMP046-07

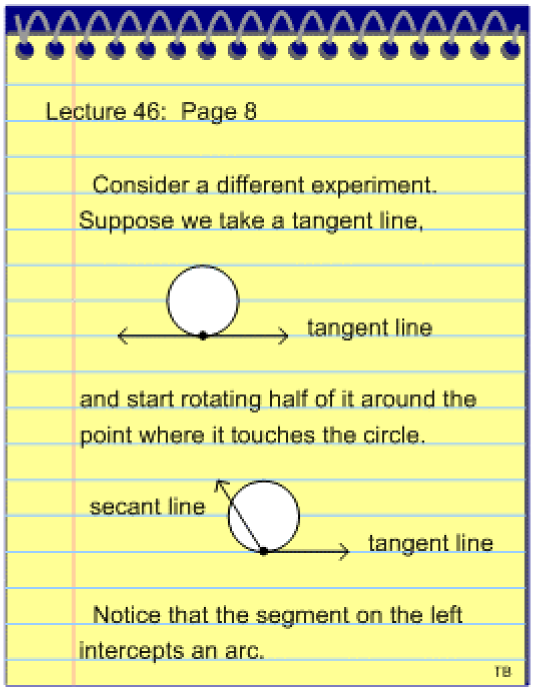


Lecture 46: Page 7

Thus, if you know the measure of the arc, then the measure of that inscribed angle is half as much.

The measure of an inscribed angle is always one-half the measure of the arc that it intercepts.

CEMP046-08



Lecture 46: Page 8

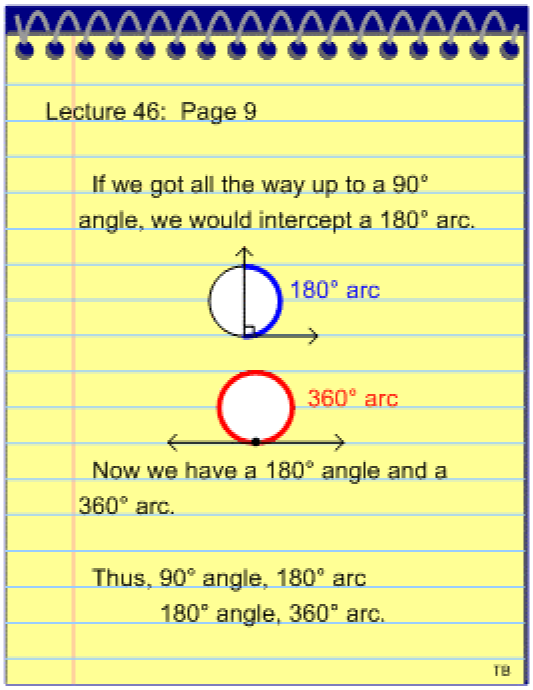
Consider a different experiment. Suppose we take a tangent line,

and start rotating half of it around the point where it touches the circle.

Notice that the segment on the left intercepts an arc.

Lecture 46 Notes, Continued

CEMP046-09



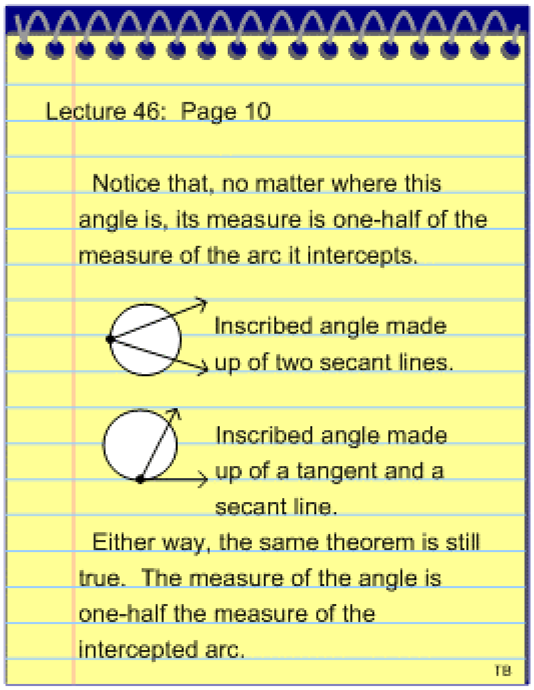
Lecture 46: Page 9

If we got all the way up to a 90° angle, we would intercept a 180° arc.

Now we have a 180° angle and a 360° arc.

Thus, 90° angle, 180° arc
180° angle, 360° arc.

CEMP046-10



Lecture 46: Page 10

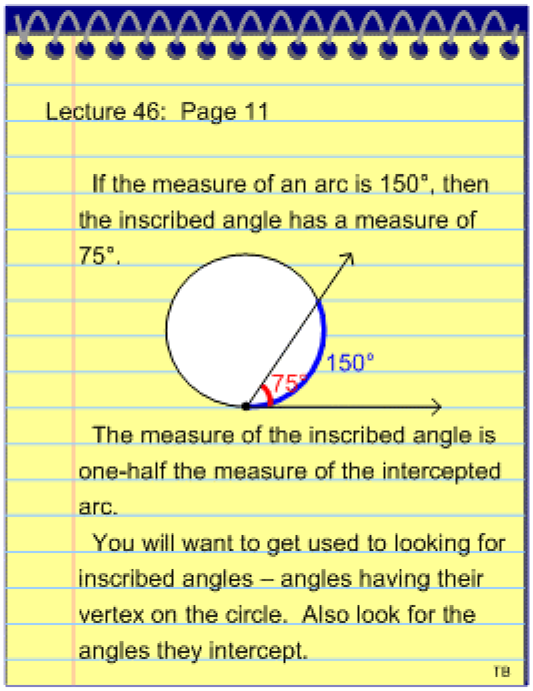
Notice that, no matter where this angle is, its measure is one-half of the measure of the arc it intercepts.

Inscribed angle made up of two secant lines.

Inscribed angle made up of a tangent and a secant line.

Either way, the same theorem is still true. The measure of the angle is one-half the measure of the intercepted arc.

CEMP046-11



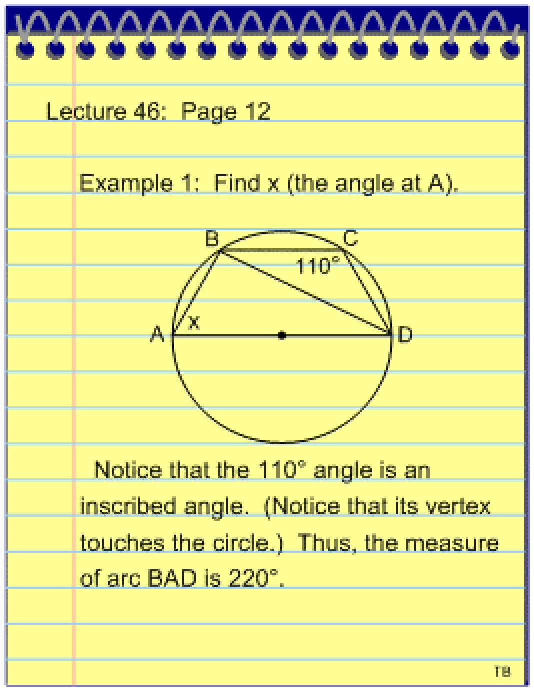
Lecture 46: Page 11

If the measure of an arc is 150°, then the inscribed angle has a measure of 75°.

The measure of the inscribed angle is one-half the measure of the intercepted arc.

You will want to get used to looking for inscribed angles – angles having their vertex on the circle. Also look for the angles they intercept.

CEMP046-12



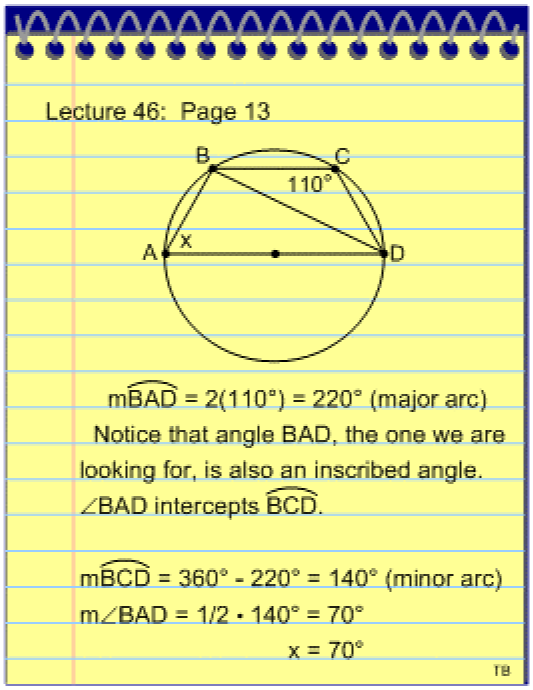
Lecture 46: Page 12

Example 1: Find x (the angle at A).

Notice that the 110° angle is an inscribed angle. (Notice that its vertex touches the circle.) Thus, the measure of arc BAD is 220°.

Lecture 46 Notes, Continued

CEMP046-13



Lecture 46: Page 13

$m\overset{\frown}{BAD} = 2(110^\circ) = 220^\circ$ (major arc)

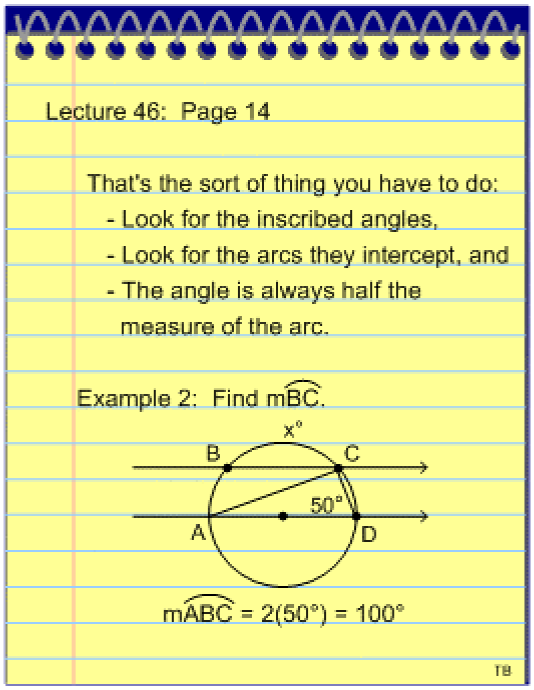
Notice that angle BAD, the one we are looking for, is also an inscribed angle. $\angle BAD$ intercepts $\overset{\frown}{BCD}$.

$m\overset{\frown}{BCD} = 360^\circ - 220^\circ = 140^\circ$ (minor arc)

$m\angle BAD = 1/2 \cdot 140^\circ = 70^\circ$

$x = 70^\circ$

CEMP046-14



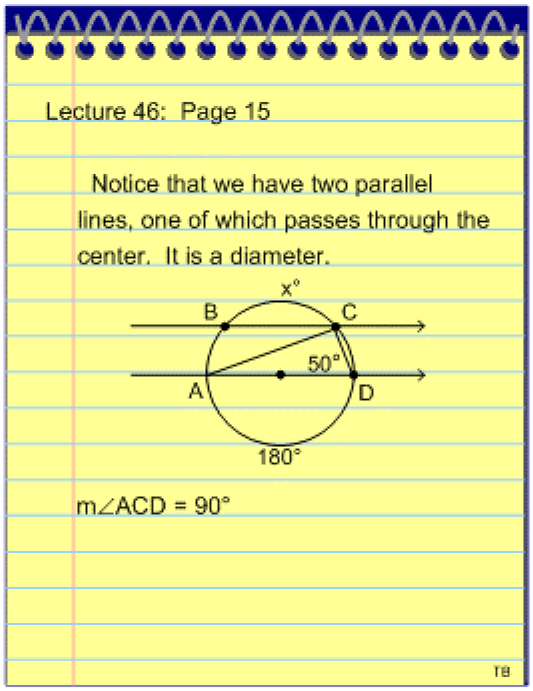
Lecture 46: Page 14

That's the sort of thing you have to do:
- Look for the inscribed angles,
- Look for the arcs they intercept, and
- The angle is always half the measure of the arc.

Example 2: Find $m\overset{\frown}{BC}$.

$m\overset{\frown}{ABC} = 2(50^\circ) = 100^\circ$

CEMP046-15

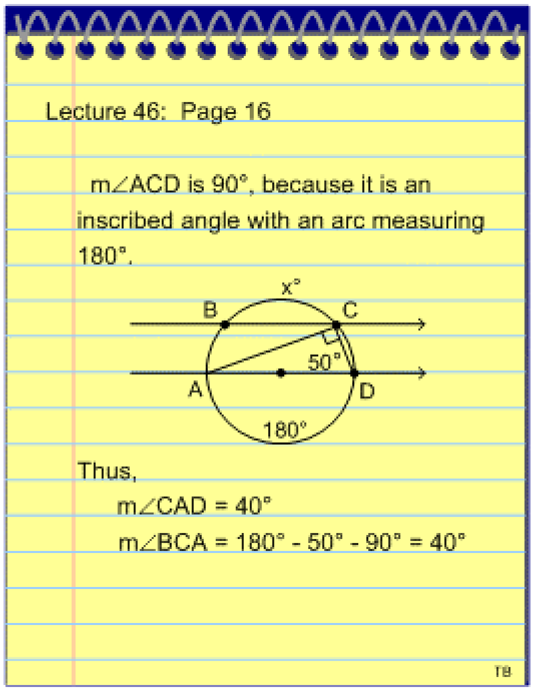


Lecture 46: Page 15

Notice that we have two parallel lines, one of which passes through the center. It is a diameter.

$m\angle ACD = 90^\circ$

CEMP046-16



Lecture 46: Page 16

$m\angle ACD$ is 90°, because it is an inscribed angle with an arc measuring 180°.

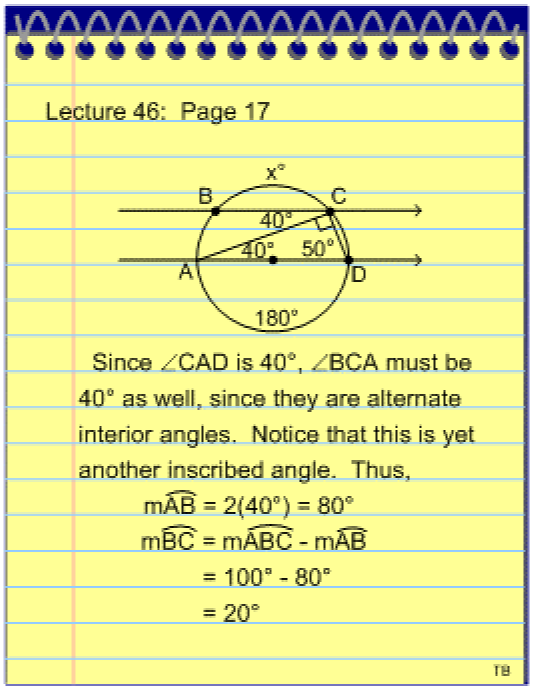
Thus,

$m\angle CAD = 40^\circ$

$m\angle BCA = 180^\circ - 50^\circ - 90^\circ = 40^\circ$

Lecture 46 Notes, Continued

CEMP046-17

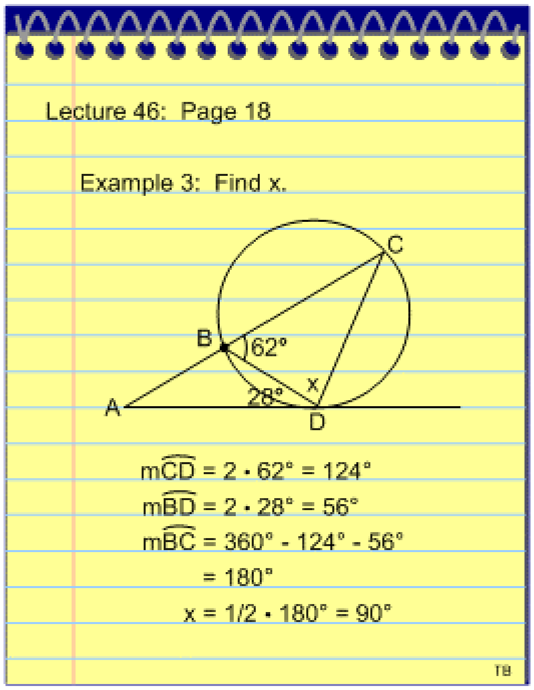


Lecture 46: Page 17

Since ∠CAD is 40°, ∠BCA must be 40° as well, since they are alternate interior angles. Notice that this is yet another inscribed angle. Thus,

$$m\widehat{AB} = 2(40°) = 80°$$
$$m\widehat{BC} = m\widehat{ABC} - m\widehat{AB}$$
$$= 100° - 80°$$
$$= 20°$$

CEMP046-18

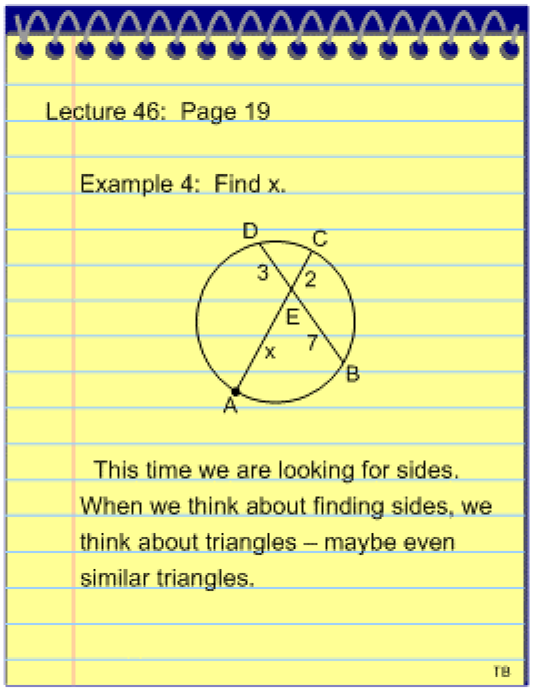


Lecture 46: Page 18

Example 3: Find x.

$$m\widehat{CD} = 2 \cdot 62° = 124°$$
$$m\widehat{BD} = 2 \cdot 28° = 56°$$
$$m\widehat{BC} = 360° - 124° - 56°$$
$$= 180°$$
$$x = 1/2 \cdot 180° = 90°$$

CEMP046-19

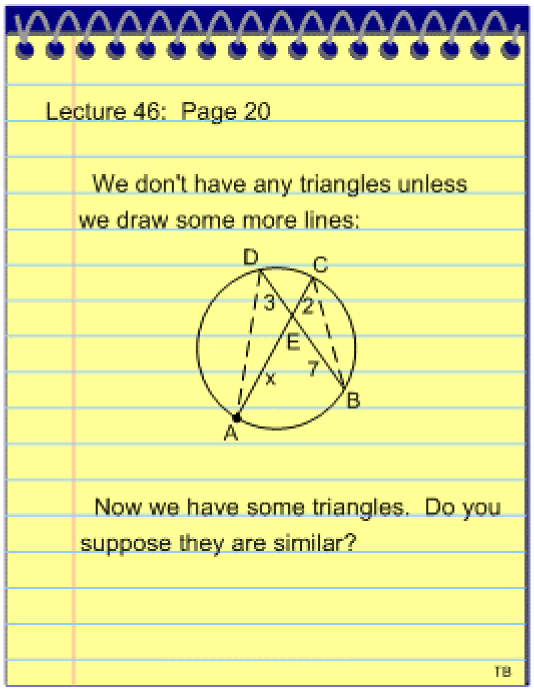


Lecture 46: Page 19

Example 4: Find x.

This time we are looking for sides. When we think about finding sides, we think about triangles – maybe even similar triangles.

CEMP046-20



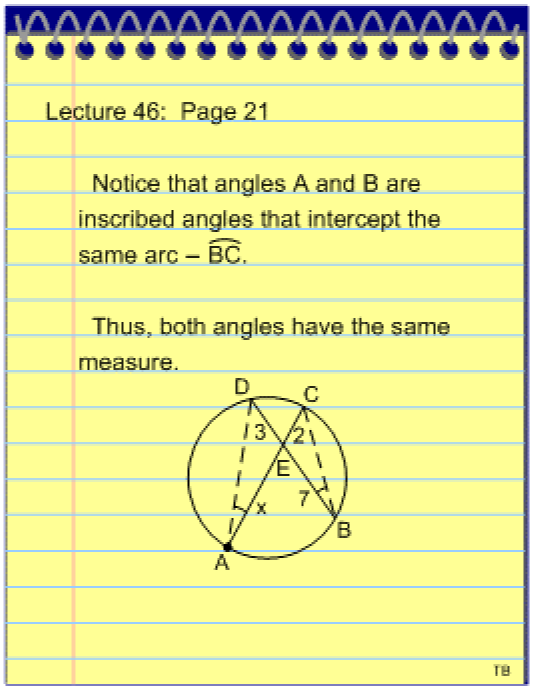
Lecture 46: Page 20

We don't have any triangles unless we draw some more lines:

Now we have some triangles. Do you suppose they are similar?

Lecture 46 Notes, Continued

CEMP046-21

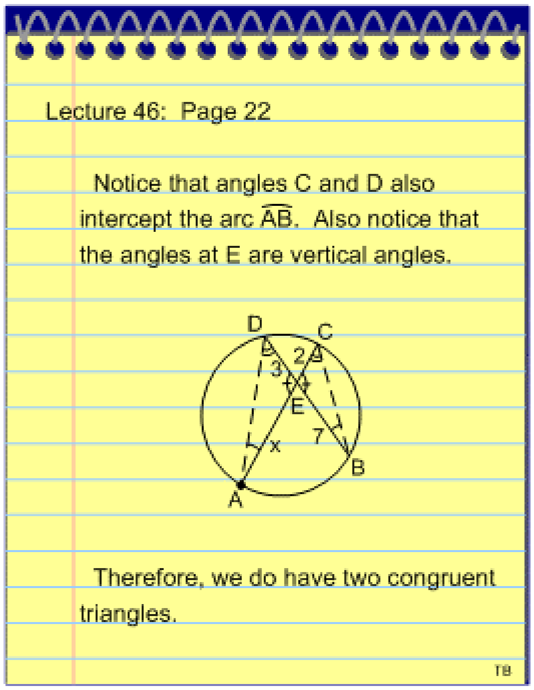


Lecture 46: Page 21

Notice that angles A and B are inscribed angles that intercept the same arc – $\widehat{BC}$.

Thus, both angles have the same measure.

CEMP046-22

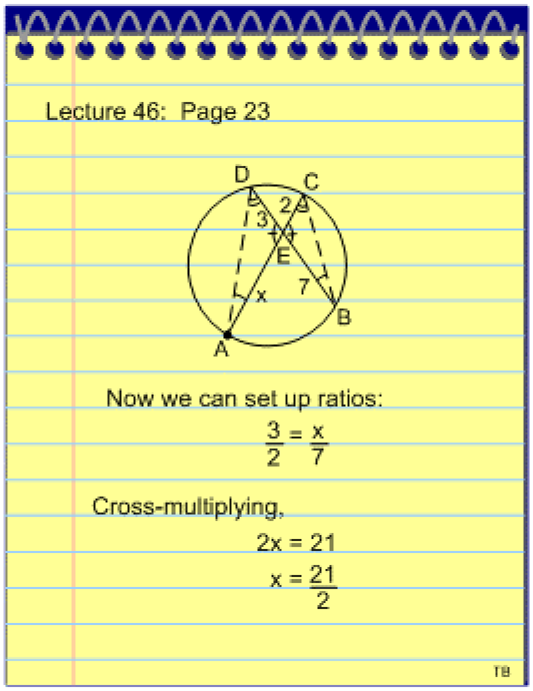


Lecture 46: Page 22

Notice that angles C and D also intercept the arc $\widehat{AB}$. Also notice that the angles at E are vertical angles.

Therefore, we do have two congruent triangles.

CEMP046-23



Lecture 46: Page 23

Now we can set up ratios:

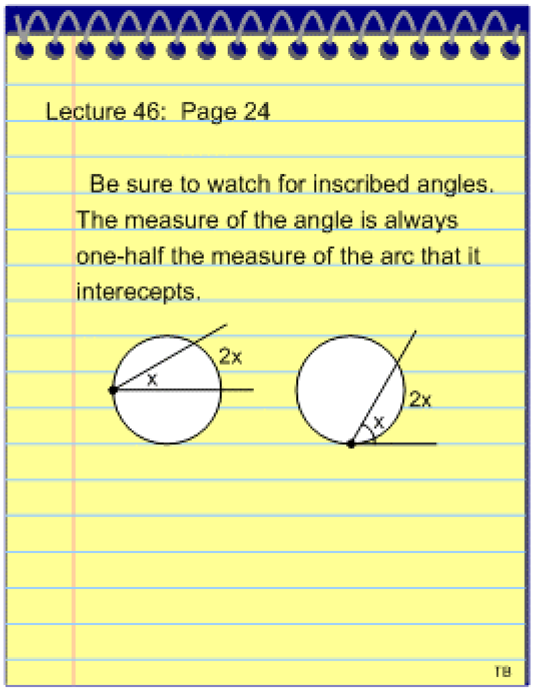
$$\frac{3}{2} = \frac{x}{7}$$

Cross-multiplying,

$$2x = 21$$

$$x = \frac{21}{2}$$

CEMP046-24

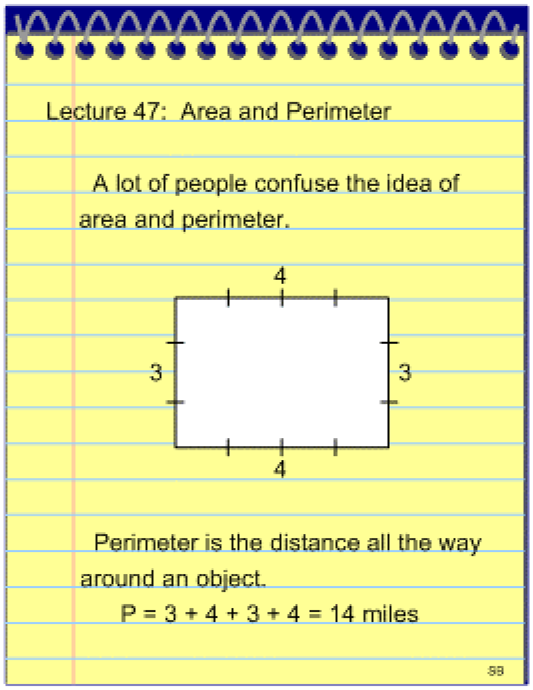


Lecture 46: Page 24

Be sure to watch for inscribed angles. The measure of the angle is always one-half the measure of the arc that it interecepts.

Lecture 47 Notes

CEMP047-01



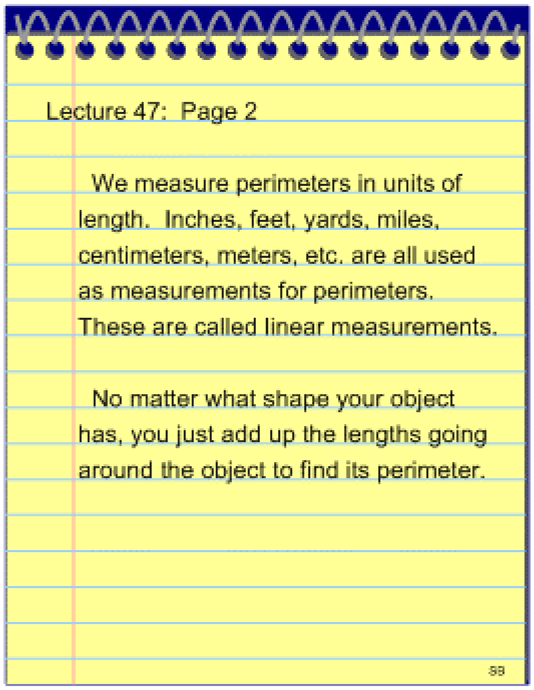
Lecture 47: Area and Perimeter

A lot of people confuse the idea of area and perimeter.

Perimeter is the distance all the way around an object.

P = 3 + 4 + 3 + 4 = 14 miles

CEMP047-02

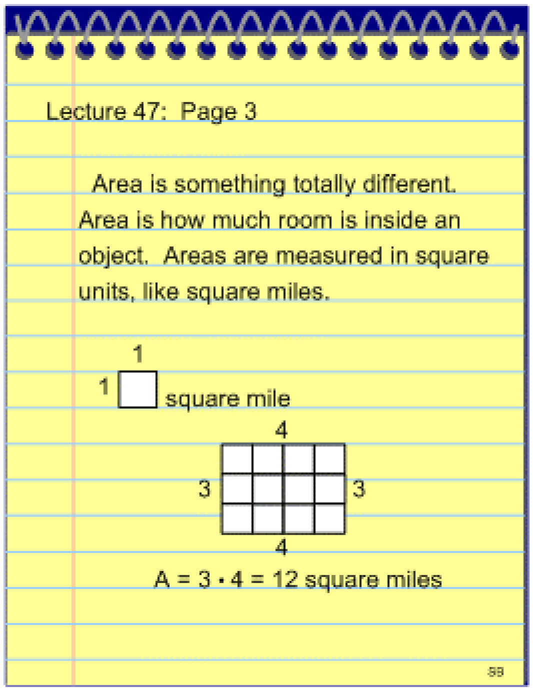


Lecture 47: Page 2

We measure perimeters in units of length. Inches, feet, yards, miles, centimeters, meters, etc. are all used as measurements for perimeters. These are called linear measurements.

No matter what shape your object has, you just add up the lengths going around the object to find its perimeter.

CEMP047-03

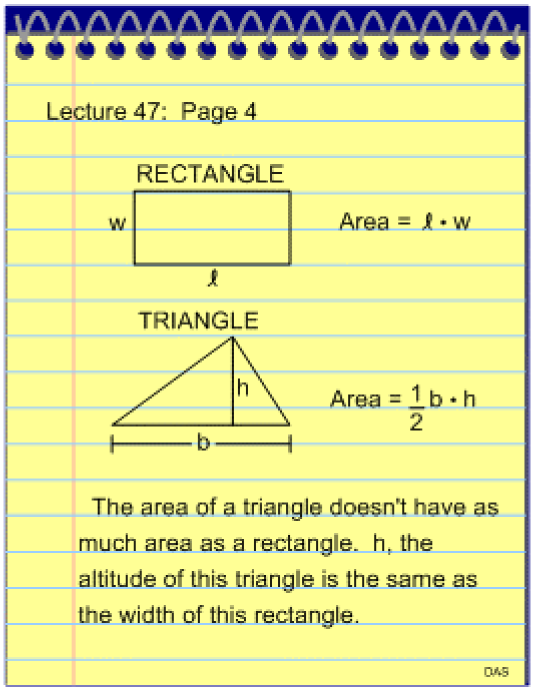


Lecture 47: Page 3

Area is something totally different. Area is how much room is inside an object. Areas are measured in square units, like square miles.

A = 3 · 4 = 12 square miles

CEMP047-04



Lecture 47: Page 4

RECTANGLE

Area = $\ell \cdot w$

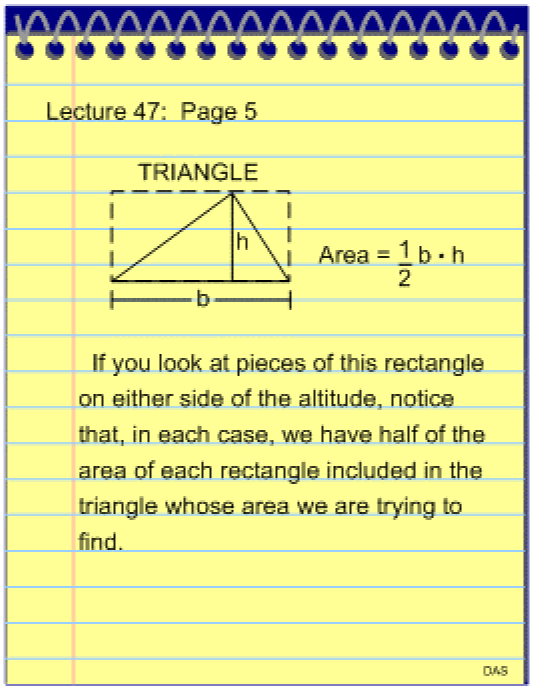
TRIANGLE

Area = $\frac{1}{2} b \cdot h$

The area of a triangle doesn't have as much area as a rectangle. h, the altitude of this triangle is the same as the width of this rectangle.

Lecture 47 Notes, Continued

CEMP047-05



Lecture 47: Page 5

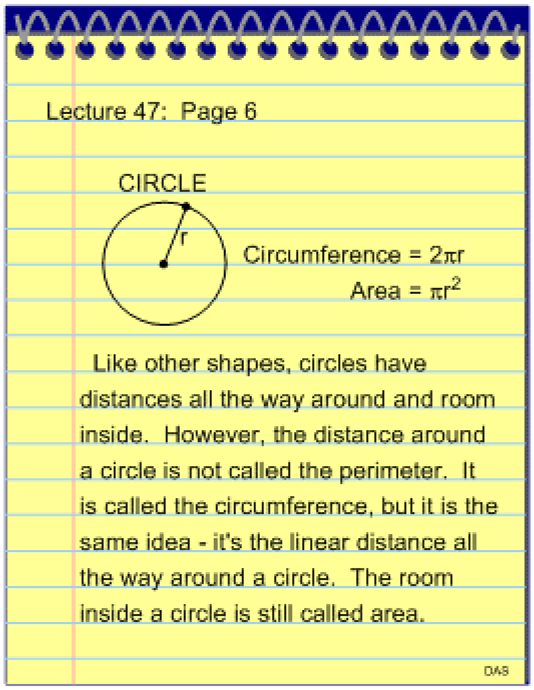
TRIANGLE

Area = $\frac{1}{2}$ b • h

If you look at pieces of this rectangle on either side of the altitude, notice that, in each case, we have half of the area of each rectangle included in the triangle whose area we are trying to find.

DAS

CEMP047-06



Lecture 47: Page 6

CIRCLE

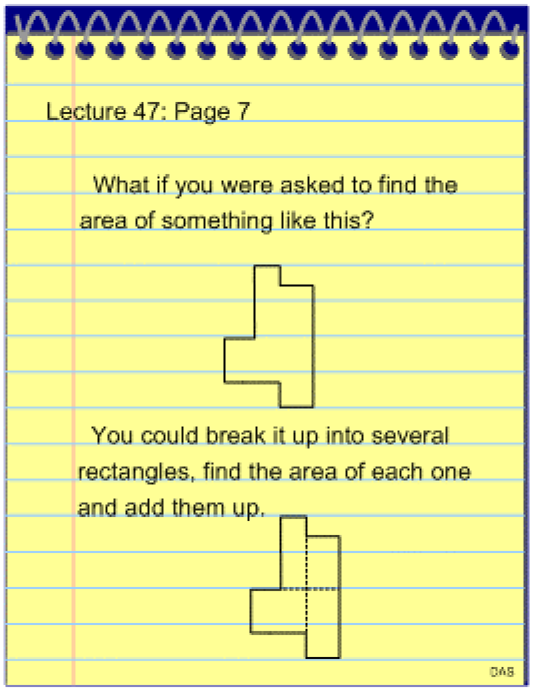
Circumference = $2\pi r$

Area = πr^2

Like other shapes, circles have distances all the way around and room inside. However, the distance around a circle is not called the perimeter. It is called the circumference, but it is the same idea - it's the linear distance all the way around a circle. The room inside a circle is still called area.

DAS

CEMP047-07



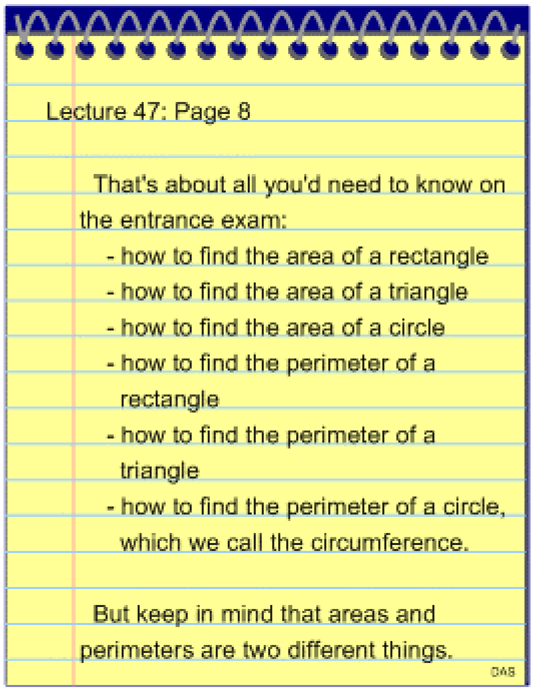
Lecture 47: Page 7

What if you were asked to find the area of something like this?

You could break it up into several rectangles, find the area of each one and add them up.

DAS

CEMP047-08



Lecture 47: Page 8

That's about all you'd need to know on the entrance exam:

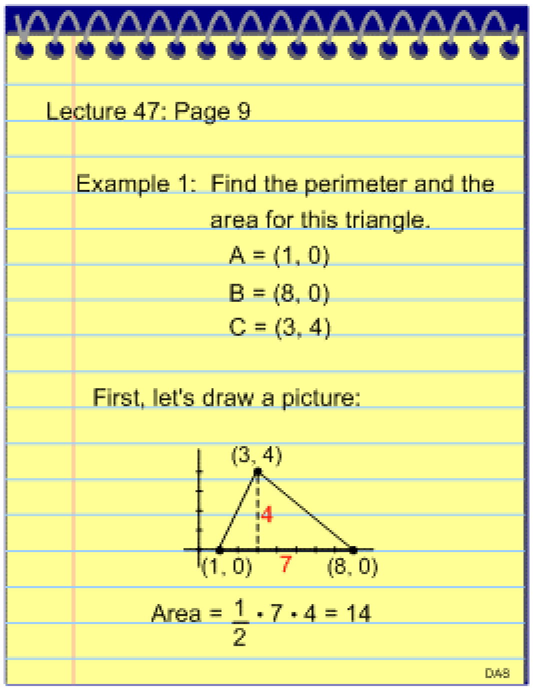
- how to find the area of a rectangle
- how to find the area of a triangle
- how to find the area of a circle
- how to find the perimeter of a rectangle
- how to find the perimeter of a triangle
- how to find the perimeter of a circle, which we call the circumference.

But keep in mind that areas and perimeters are two different things.

DAS

Lecture 47 Notes, Continued

CEMP047-09



Lecture 47: Page 9

Example 1: Find the perimeter and the area for this triangle.

A = (1, 0)

B = (8, 0)

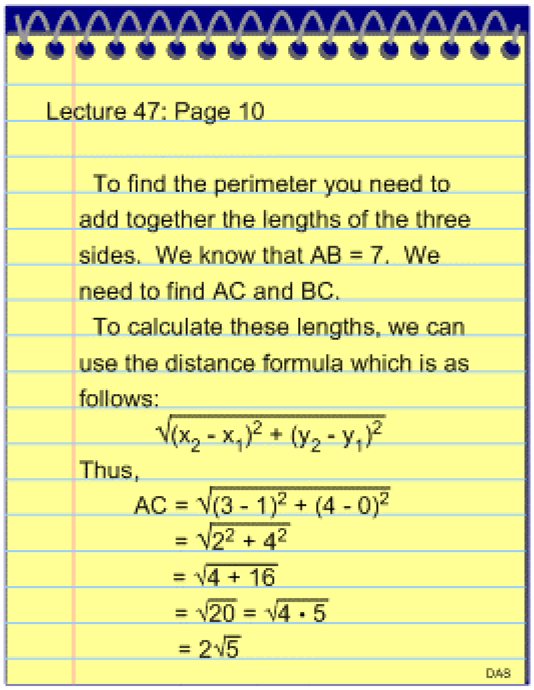
C = (3, 4)

First, let's draw a picture:

$$\text{Area} = \frac{1}{2} \cdot 7 \cdot 4 = 14$$

DAS

CEMP047-10



Lecture 47: Page 10

To find the perimeter you need to add together the lengths of the three sides. We know that AB = 7. We need to find AC and BC.

To calculate these lengths, we can use the distance formula which is as follows:

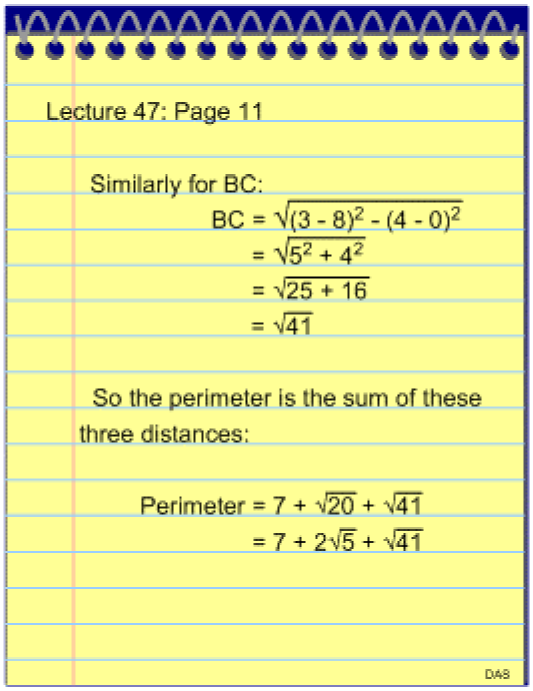
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Thus,

$$\begin{aligned} AC &= \sqrt{(3 - 1)^2 + (4 - 0)^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} = \sqrt{4 \cdot 5} \\ &= 2\sqrt{5} \end{aligned}$$

DAS

CEMP047-11



Lecture 47: Page 11

Similarly for BC:

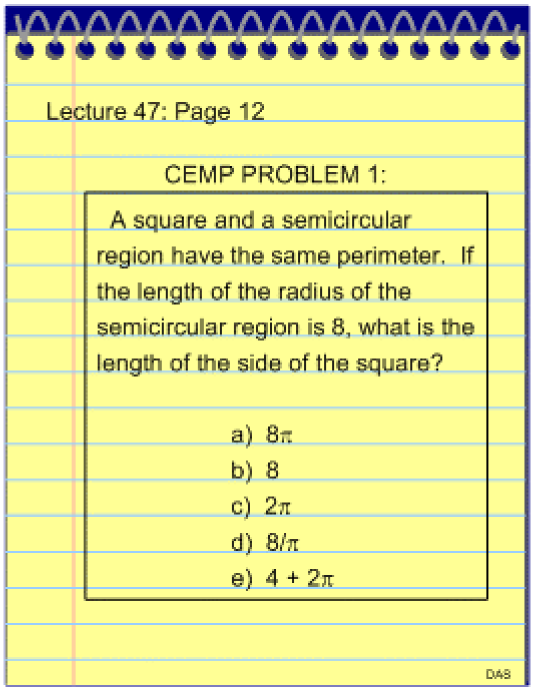
$$\begin{aligned} BC &= \sqrt{(3 - 8)^2 - (4 - 0)^2} \\ &= \sqrt{5^2 + 4^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

So the perimeter is the sum of these three distances:

$$\begin{aligned} \text{Perimeter} &= 7 + \sqrt{20} + \sqrt{41} \\ &= 7 + 2\sqrt{5} + \sqrt{41} \end{aligned}$$

DAS

CEMP047-12



Lecture 47: Page 12

CEMP PROBLEM 1:

A square and a semicircular region have the same perimeter. If the length of the radius of the semicircular region is 8, what is the length of the side of the square?

a) 8π

b) 8

c) 2π

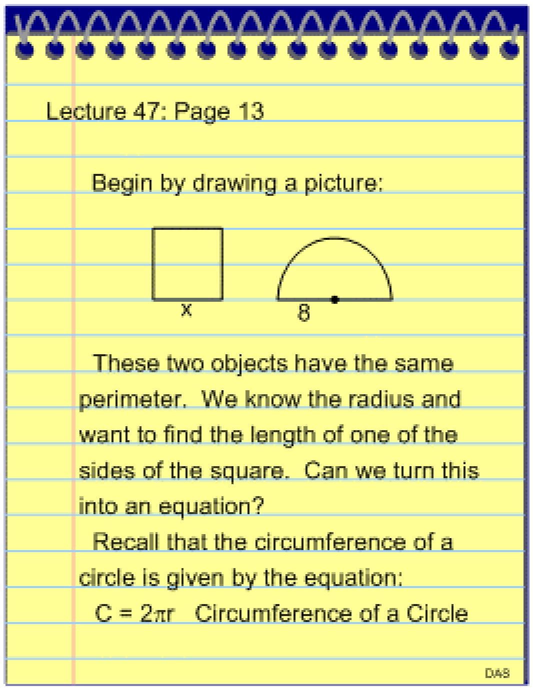
d) $8/\pi$

e) $4 + 2\pi$

DAS

Lecture 47 Notes, Continued

CEMP047-13



Lecture 47: Page 13

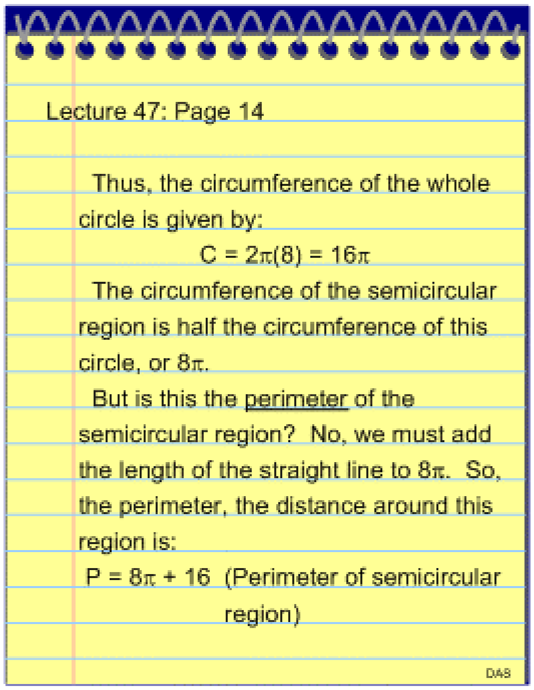
Begin by drawing a picture:

These two objects have the same perimeter. We know the radius and want to find the length of one of the sides of the square. Can we turn this into an equation?

Recall that the circumference of a circle is given by the equation:

$C = 2\pi r$ Circumference of a Circle

CEMP047-14



Lecture 47: Page 14

Thus, the circumference of the whole circle is given by:

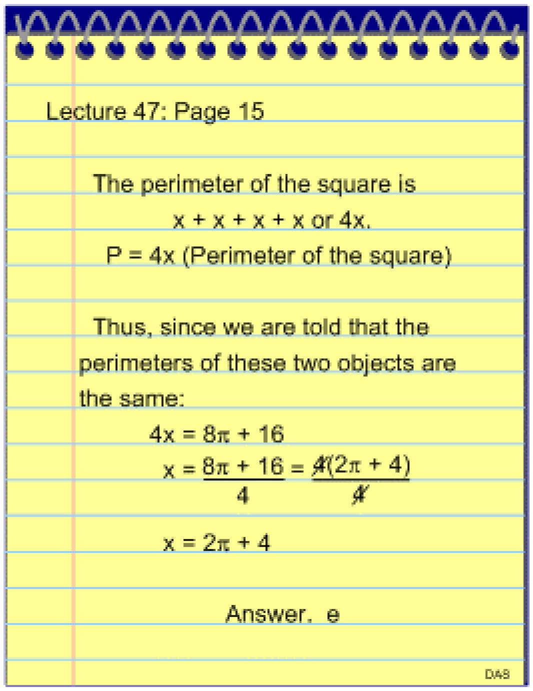
$$C = 2\pi(8) = 16\pi$$

The circumference of the semicircular region is half the circumference of this circle, or 8π.

But is this the <u>perimeter</u> of the semicircular region? No, we must add the length of the straight line to 8π. So, the perimeter, the distance around this region is:

$P = 8\pi + 16$ (Perimeter of semicircular region)

CEMP047-15



Lecture 47: Page 15

The perimeter of the square is $x + x + x + x$ or $4x$.

$P = 4x$ (Perimeter of the square)

Thus, since we are told that the perimeters of these two objects are the same:

$$4x = 8\pi + 16$$

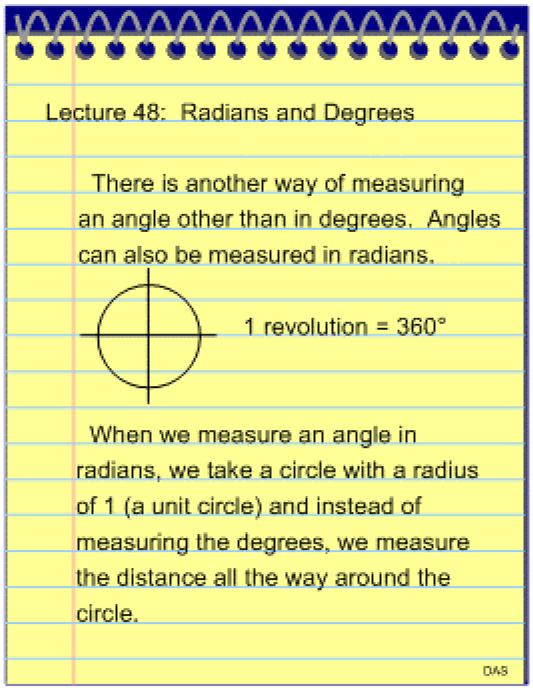
$$x = \frac{8\pi + 16}{4} = \frac{\cancel{4}(2\pi + 4)}{\cancel{4}}$$

$$x = 2\pi + 4$$

Answer. e

Lecture 48 Notes

CEMP048-01



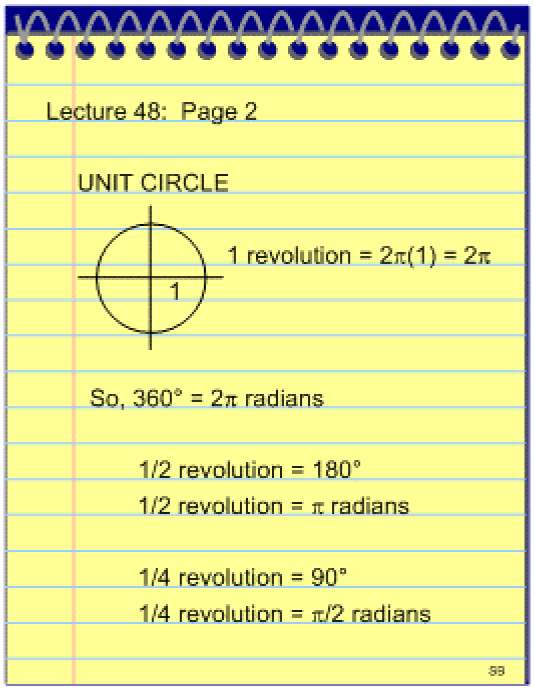
Lecture 48: Radians and Degrees

There is another way of measuring an angle other than in degrees. Angles can also be measured in radians.

1 revolution = 360°

When we measure an angle in radians, we take a circle with a radius of 1 (a unit circle) and instead of measuring the degrees, we measure the distance all the way around the circle.

CEMP048-02



Lecture 48: Page 2

UNIT CIRCLE

1 revolution = $2\pi(1) = 2\pi$

So, 360° = 2π radians

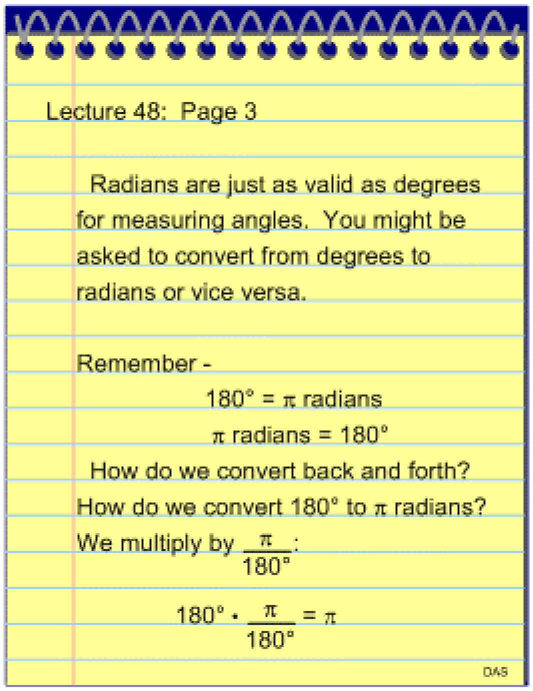
1/2 revolution = 180°

1/2 revolution = π radians

1/4 revolution = 90°

1/4 revolution = $\pi/2$ radians

CEMP048-03



Lecture 48: Page 3

Radians are just as valid as degrees for measuring angles. You might be asked to convert from degrees to radians or vice versa.

Remember -

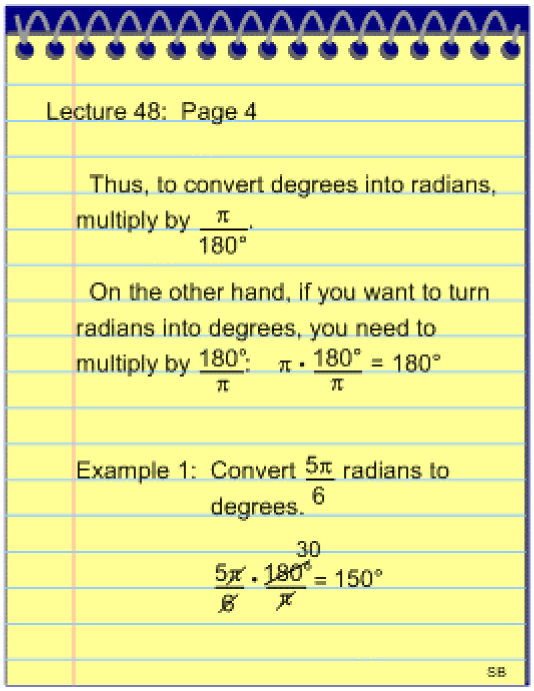
180° = π radians

π radians = 180°

How do we convert back and forth? How do we convert 180° to π radians? We multiply by $\frac{\pi}{180^\circ}$:

$$180^\circ \cdot \frac{\pi}{180^\circ} = \pi$$

CEMP048-04



Lecture 48: Page 4

Thus, to convert degrees into radians, multiply by $\frac{\pi}{180^\circ}$.

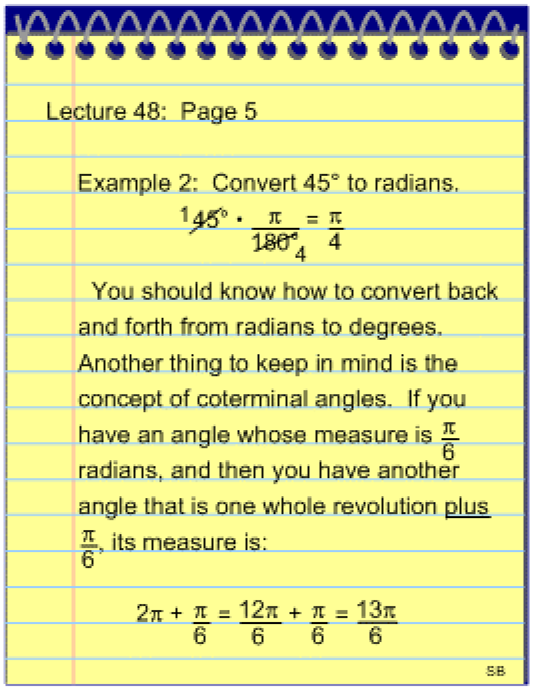
On the other hand, if you want to turn radians into degrees, you need to multiply by $\frac{180^\circ}{\pi}$: $\pi \cdot \frac{180^\circ}{\pi} = 180^\circ$

Example 1: Convert $\frac{5\pi}{6}$ radians to degrees.

$$\frac{5\cancel{\pi}}{\cancel{6}} \cdot \frac{\overset{30}{\cancel{180^\circ}}}{\cancel{\pi}} = 150^\circ$$

Lecture 48 Notes, Continued

CEMP048-05



Lecture 48: Page 5

Example 2: Convert 45° to radians.

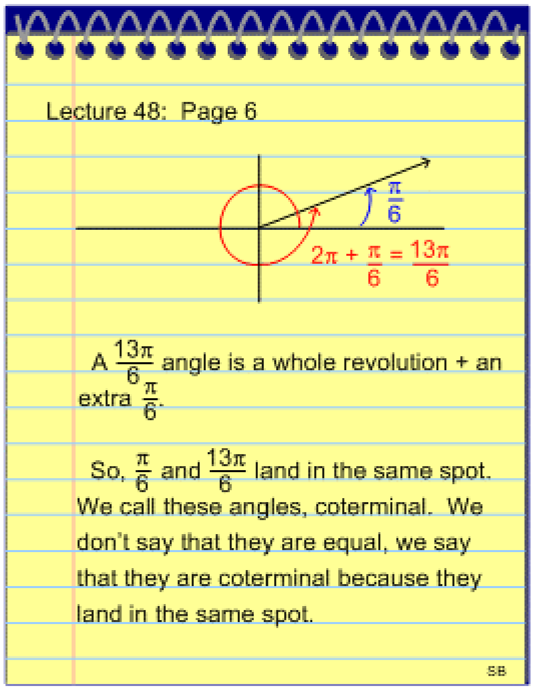
$$\overset{1}{\cancel{45°}} \cdot \frac{\pi}{\cancel{180°}_{4}} = \frac{\pi}{4}$$

You should know how to convert back and forth from radians to degrees. Another thing to keep in mind is the concept of coterminal angles. If you have an angle whose measure is $\frac{\pi}{6}$ radians, and then you have another angle that is one whole revolution plus $\frac{\pi}{6}$, its measure is:

$$2\pi + \frac{\pi}{6} = \frac{12\pi}{6} + \frac{\pi}{6} = \frac{13\pi}{6}$$

SB

CEMP048-06



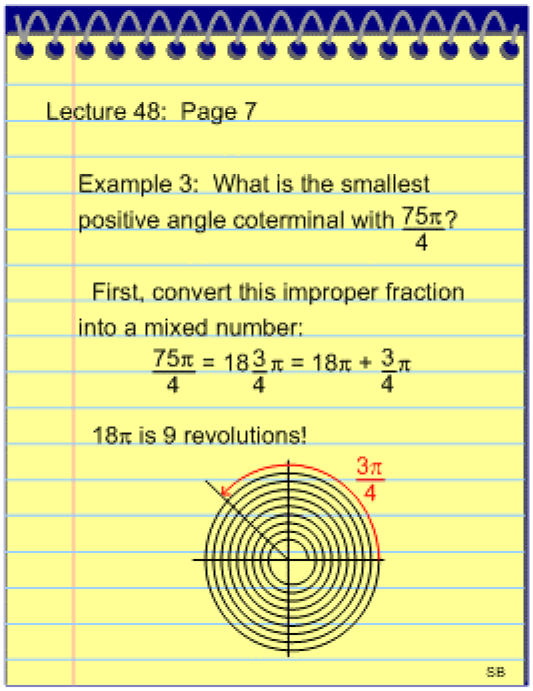
Lecture 48: Page 6

A $\frac{13\pi}{6}$ angle is a whole revolution + an extra $\frac{\pi}{6}$.

So, $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ land in the same spot. We call these angles, coterminal. We don't say that they are equal, we say that they are coterminal because they land in the same spot.

SB

CEMP048-07



Lecture 48: Page 7

Example 3: What is the smallest positive angle coterminal with $\frac{75\pi}{4}$?

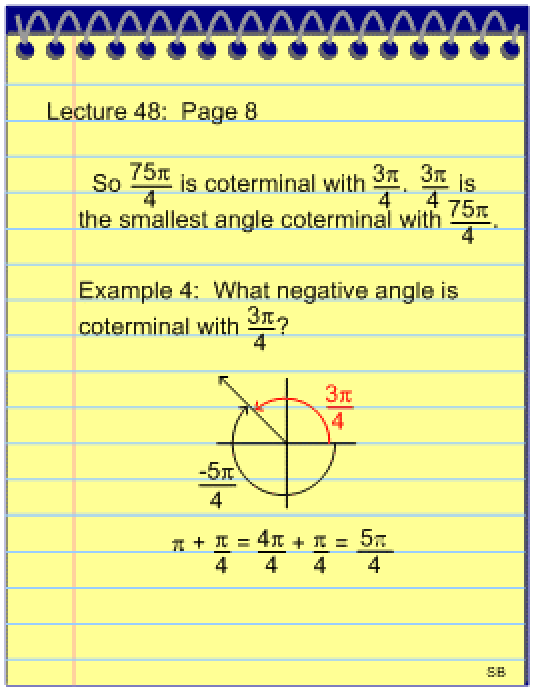
First, convert this improper fraction into a mixed number:

$$\frac{75\pi}{4} = 18\frac{3}{4}\pi = 18\pi + \frac{3}{4}\pi$$

18π is 9 revolutions!

SB

CEMP048-08



Lecture 48: Page 8

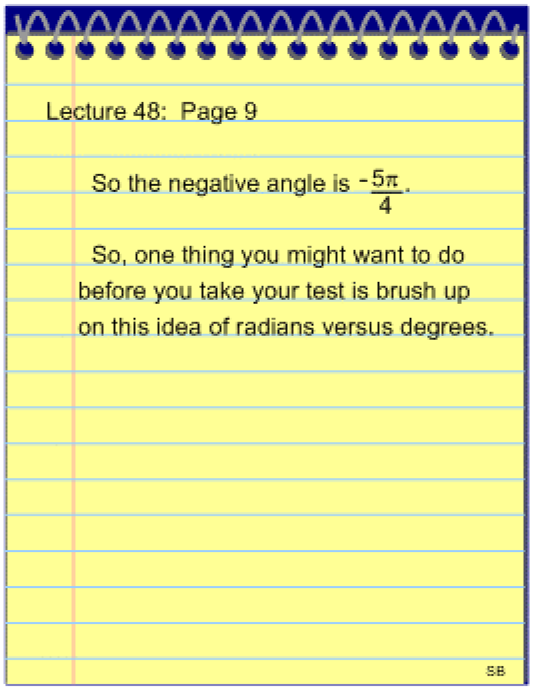
So $\frac{75\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. $\frac{3\pi}{4}$ is the smallest angle coterminal with $\frac{75\pi}{4}$.

Example 4: What negative angle is coterminal with $\frac{3\pi}{4}$?

$$\pi + \frac{\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4} = \frac{5\pi}{4}$$

SB

CEMP048-09



Lecture 48: Page 9

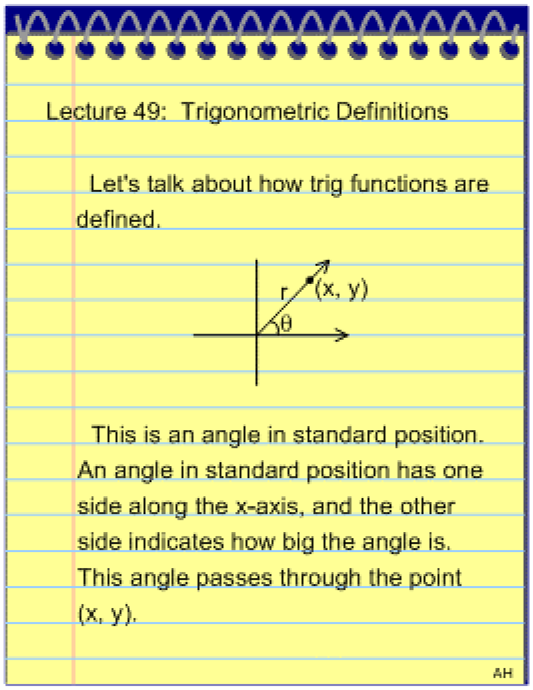
So the negative angle is $-\frac{5\pi}{4}$.

So, one thing you might want to do before you take your test is brush up on this idea of radians versus degrees.

SB

Lecture 49 Notes

CEMP049-01



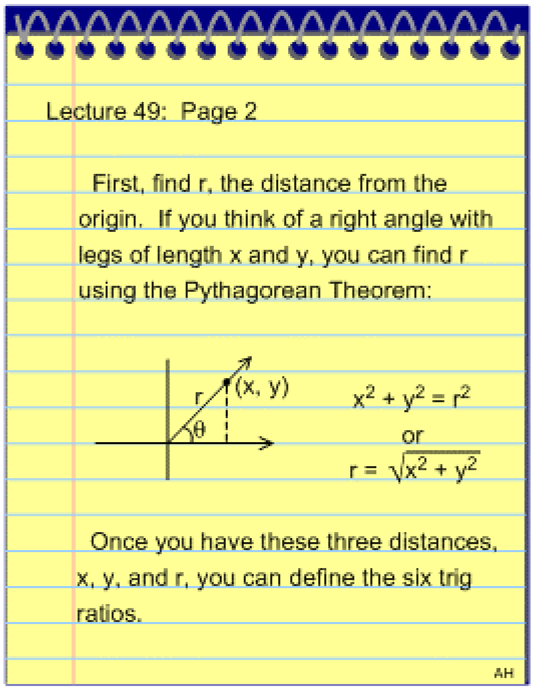
Lecture 49: Trigonometric Definitions

Let's talk about how trig functions are defined.

This is an angle in standard position. An angle in standard position has one side along the x-axis, and the other side indicates how big the angle is. This angle passes through the point (x, y).

AH

CEMP049-02



Lecture 49: Page 2

First, find r, the distance from the origin. If you think of a right angle with legs of length x and y, you can find r using the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

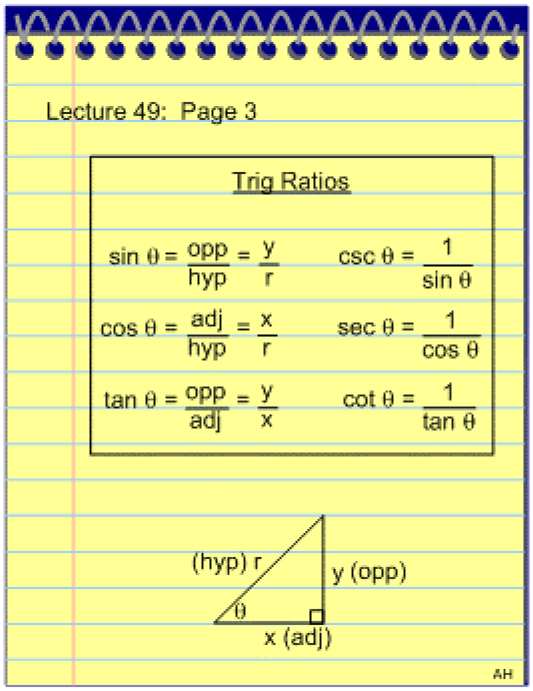
or

$$r = \sqrt{x^2 + y^2}$$

Once you have these three distances, x, y, and r, you can define the six trig ratios.

AH

CEMP049-03



Lecture 49: Page 3

Trig Ratios

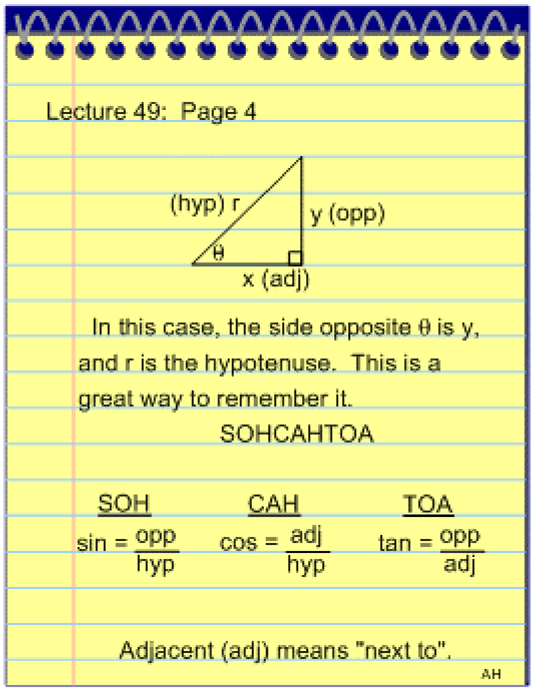
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta}$$

AH

CEMP049-04



Lecture 49: Page 4

In this case, the side opposite θ is y, and r is the hypotenuse. This is a great way to remember it.

SOHCAHTOA

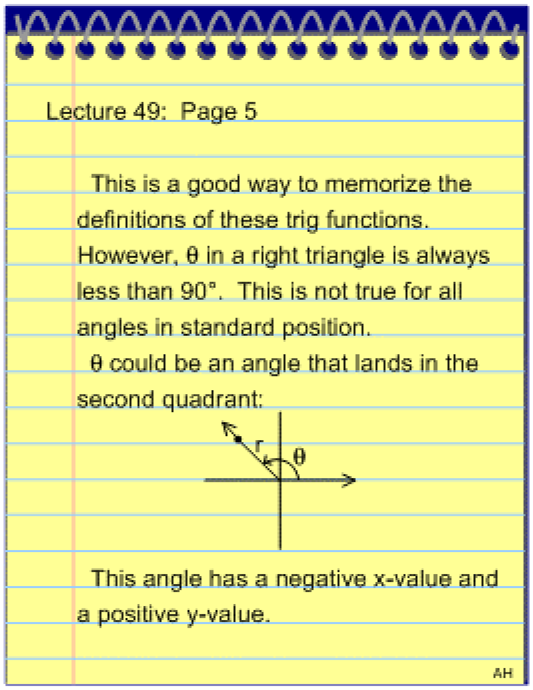
SOH	CAH	TOA
$\sin = \frac{\text{opp}}{\text{hyp}}$	$\cos = \frac{\text{adj}}{\text{hyp}}$	$\tan = \frac{\text{opp}}{\text{adj}}$

Adjacent (adj) means "next to".

AH

Lecture 49 Notes, Continued

CEMP049-05



Lecture 49: Page 5

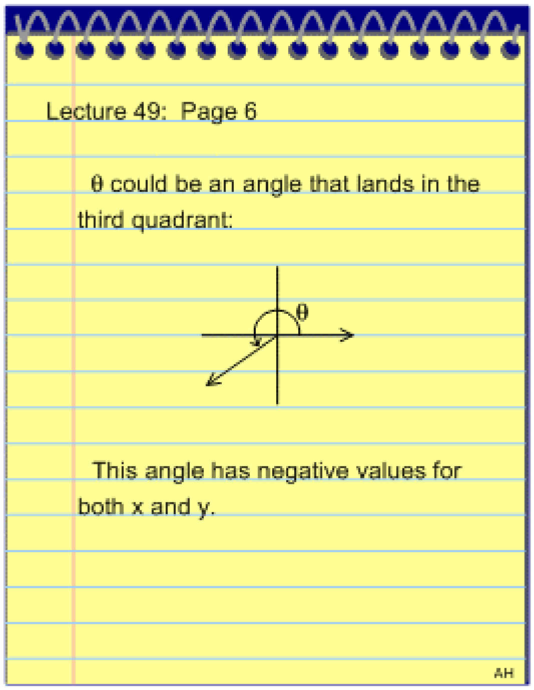
This is a good way to memorize the definitions of these trig functions. However, θ in a right triangle is always less than 90°. This is not true for all angles in standard position.

θ could be an angle that lands in the second quadrant:

This angle has a negative x-value and a positive y-value.

AH

CEMP049-06



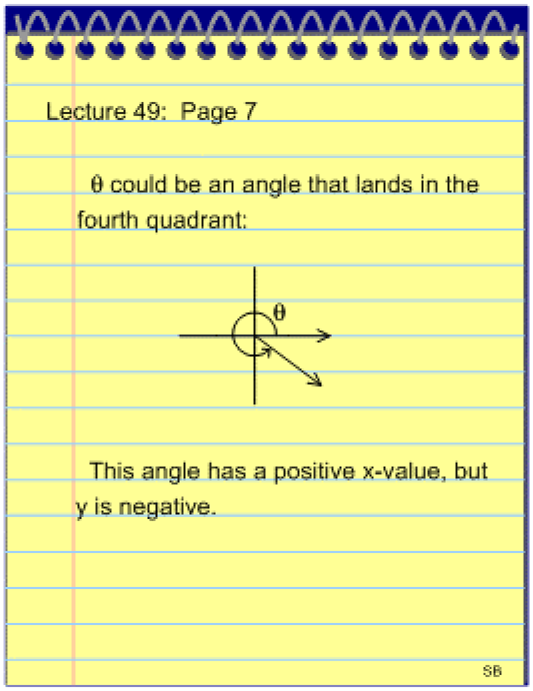
Lecture 49: Page 6

θ could be an angle that lands in the third quadrant:

This angle has negative values for both x and y.

AH

CEMP049-07



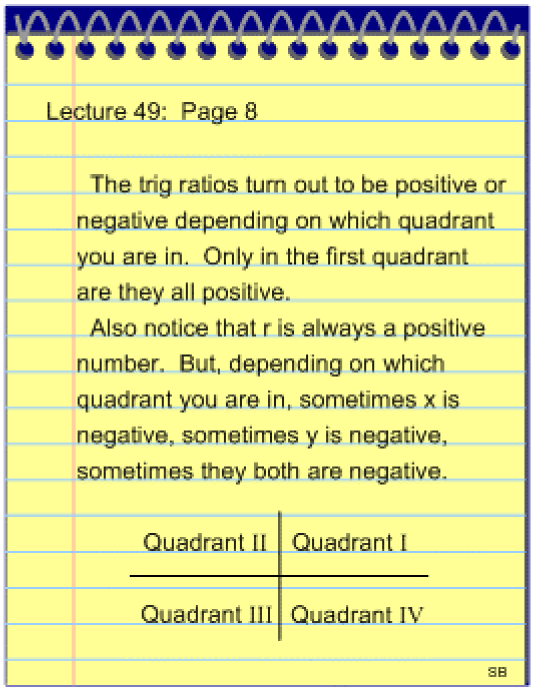
Lecture 49: Page 7

θ could be an angle that lands in the fourth quadrant:

This angle has a positive x-value, but y is negative.

SB

CEMP049-08



Lecture 49: Page 8

The trig ratios turn out to be positive or negative depending on which quadrant you are in. Only in the first quadrant are they all positive.

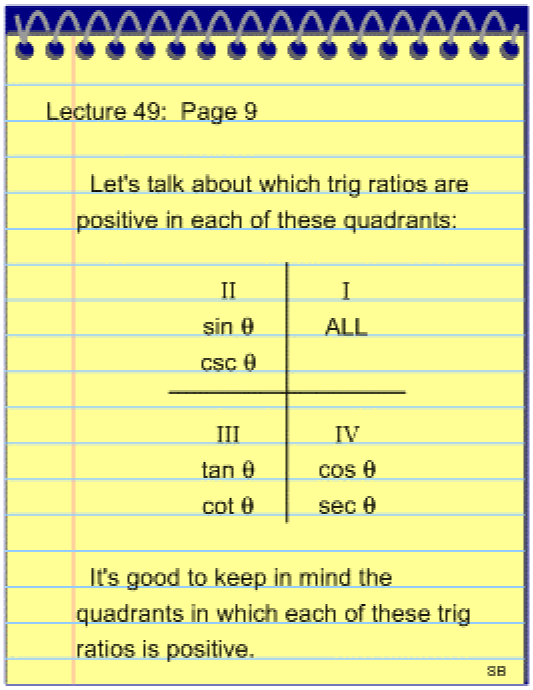
Also notice that r is always a positive number. But, depending on which quadrant you are in, sometimes x is negative, sometimes y is negative, sometimes they both are negative.

Quadrant II	Quadrant I
Quadrant III	Quadrant IV

SB

Lecture 49 Notes, Continued

CEMP049-09



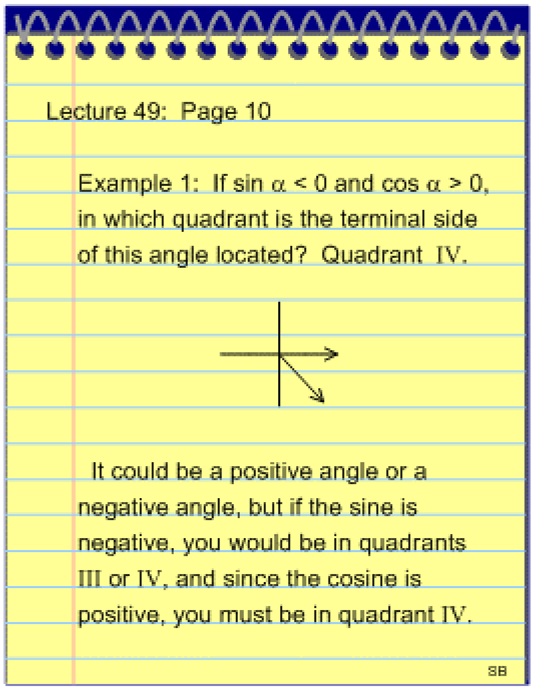
Lecture 49: Page 9

Let's talk about which trig ratios are positive in each of these quadrants:

II sin θ csc θ	I ALL
III tan θ cot θ	IV cos θ sec θ

It's good to keep in mind the quadrants in which each of these trig ratios is positive.

CEMP049-10

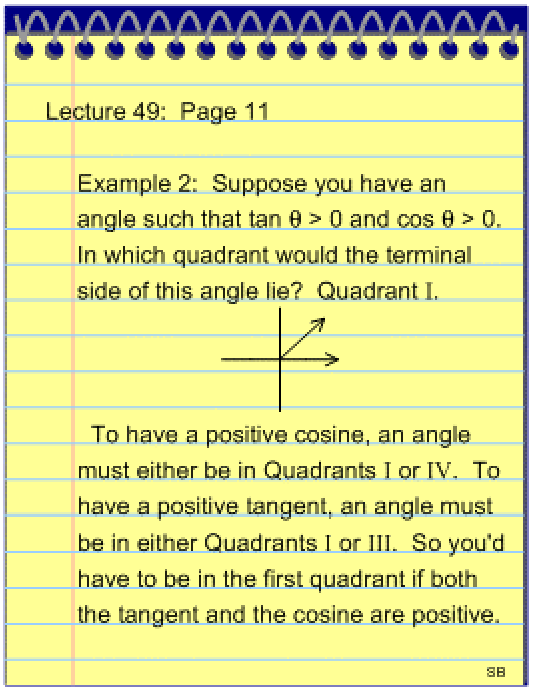


Lecture 49: Page 10

Example 1: If sin $\alpha < 0$ and cos $\alpha > 0$, in which quadrant is the terminal side of this angle located? Quadrant IV.

It could be a positive angle or a negative angle, but if the sine is negative, you would be in quadrants III or IV, and since the cosine is positive, you must be in quadrant IV.

CEMP049-11

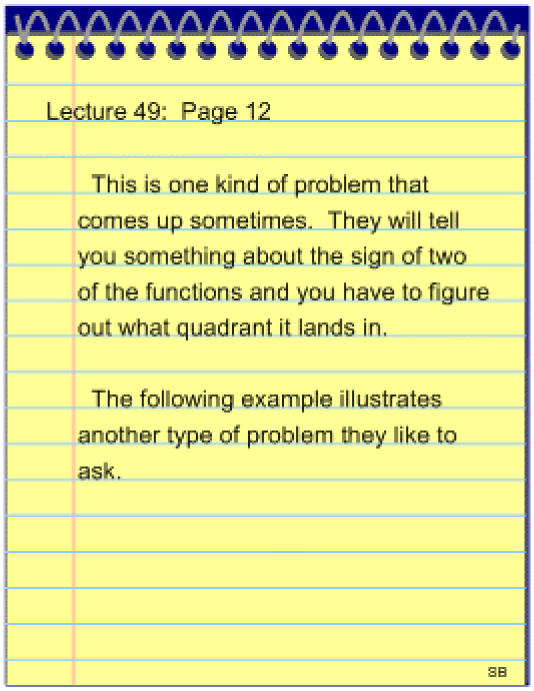


Lecture 49: Page 11

Example 2: Suppose you have an angle such that tan $\theta > 0$ and cos $\theta > 0$. In which quadrant would the terminal side of this angle lie? Quadrant I.

To have a positive cosine, an angle must either be in Quadrants I or IV. To have a positive tangent, an angle must be in either Quadrants I or III. So you'd have to be in the first quadrant if both the tangent and the cosine are positive.

CEMP049-12



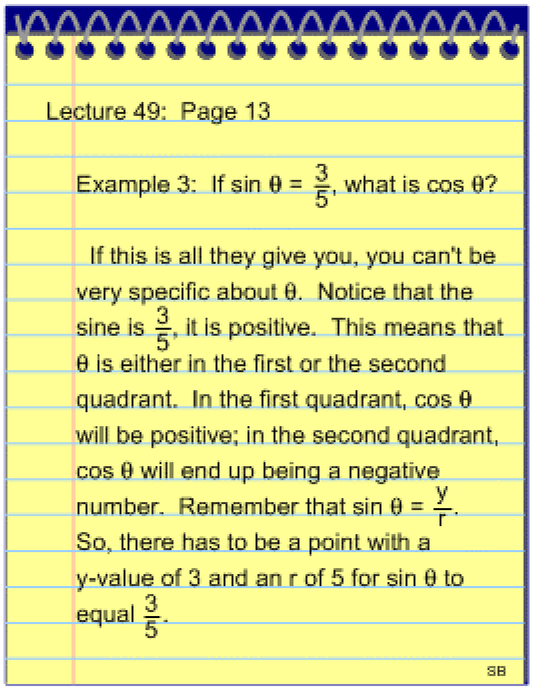
Lecture 49: Page 12

This is one kind of problem that comes up sometimes. They will tell you something about the sign of two of the functions and you have to figure out what quadrant it lands in.

The following example illustrates another type of problem they like to ask.

Lecture 49 Notes, Continued

CEMP049-13



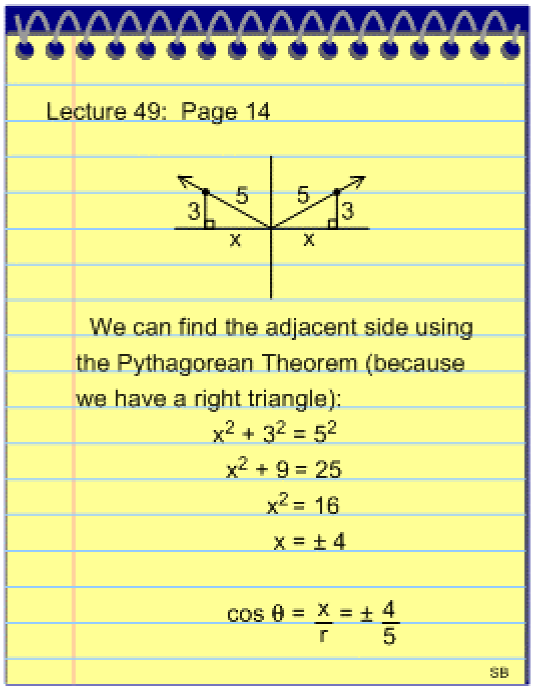
Lecture 49: Page 13

Example 3: If sin θ = $\frac{3}{5}$, what is cos θ?

If this is all they give you, you can't be very specific about θ. Notice that the sine is $\frac{3}{5}$, it is positive. This means that θ is either in the first or the second quadrant. In the first quadrant, cos θ will be positive; in the second quadrant, cos θ will end up being a negative number. Remember that sin θ = $\frac{y}{r}$. So, there has to be a point with a y-value of 3 and an r of 5 for sin θ to equal $\frac{3}{5}$.

SB

CEMP049-14



Lecture 49: Page 14

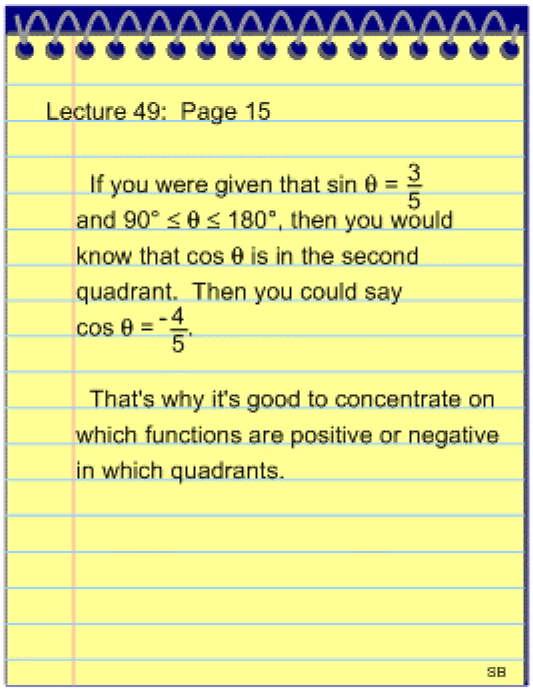
We can find the adjacent side using the Pythagorean Theorem (because we have a right triangle):

$$x^2 + 3^2 = 5^2$$
$$x^2 + 9 = 25$$
$$x^2 = 16$$
$$x = \pm 4$$

$$\cos\theta = \frac{x}{r} = \pm\frac{4}{5}$$

SB

CEMP049-15



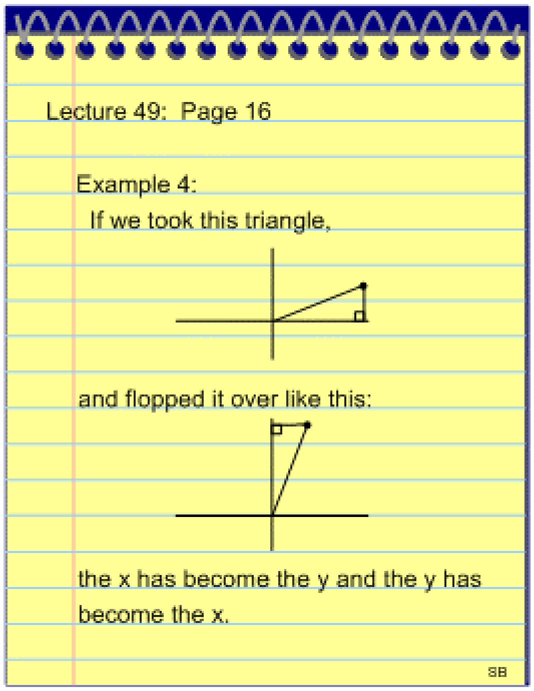
Lecture 49: Page 15

If you were given that sin θ = $\frac{3}{5}$ and 90° ≤ θ ≤ 180°, then you would know that cos θ is in the second quadrant. Then you could say cos θ = $\frac{-4}{5}$.

That's why it's good to concentrate on which functions are positive or negative in which quadrants.

SB

CEMP049-16



Lecture 49: Page 16

Example 4:

If we took this triangle,

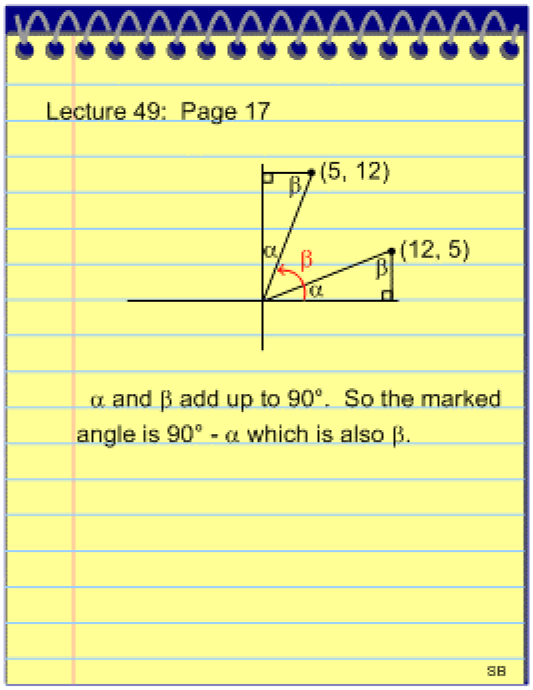
and flopped it over like this:

the x has become the y and the y has become the x.

SB

Lecture 49 Notes, Continued

CEMP049-17

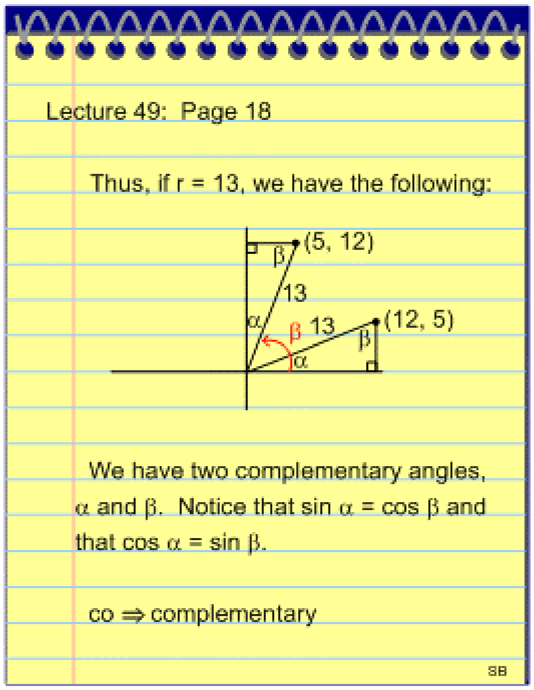


Lecture 49: Page 17

α and β add up to 90°. So the marked angle is 90° - α which is also β.

SB

CEMP049-18



Lecture 49: Page 18

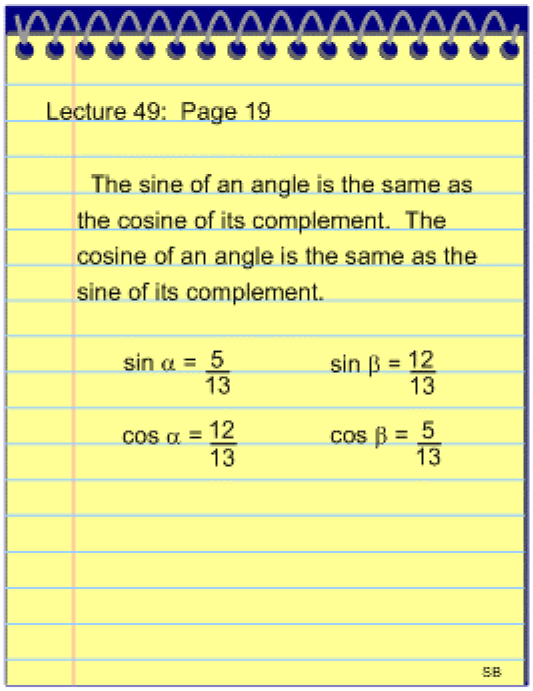
Thus, if r = 13, we have the following:

We have two complementary angles, α and β. Notice that sin α = cos β and that cos α = sin β.

co ⇒ complementary

SB

CEMP049-19



Lecture 49: Page 19

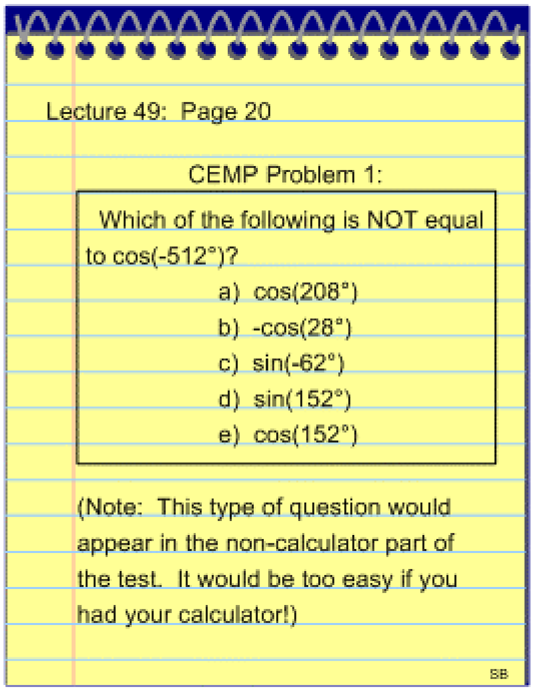
The sine of an angle is the same as the cosine of its complement. The cosine of an angle is the same as the sine of its complement.

$\sin\alpha = \frac{5}{13}$　　$\sin\beta = \frac{12}{13}$

$\cos\alpha = \frac{12}{13}$　　$\cos\beta = \frac{5}{13}$

SB

CEMP049-20



Lecture 49: Page 20

CEMP Problem 1:

Which of the following is NOT equal to cos(-512°)?

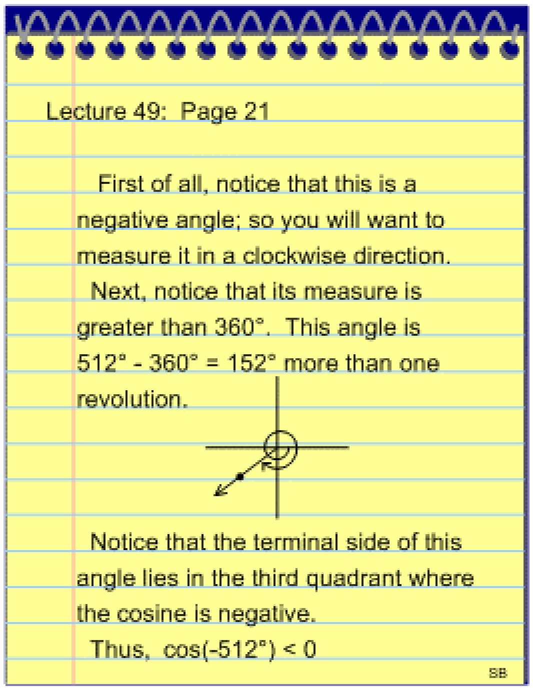
a) cos(208°)
b) -cos(28°)
c) sin(-62°)
d) sin(152°)
e) cos(152°)

(Note: This type of question would appear in the non-calculator part of the test. It would be too easy if you had your calculator!)

SB

Lecture 49 Notes, Continued

CEMP049-21



Lecture 49: Page 21

First of all, notice that this is a negative angle; so you will want to measure it in a clockwise direction.

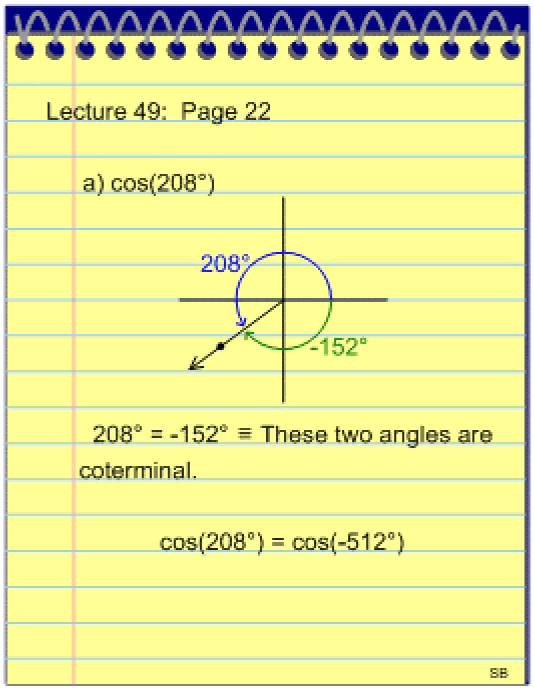
Next, notice that its measure is greater than 360°. This angle is 512° - 360° = 152° more than one revolution.

Notice that the terminal side of this angle lies in the third quadrant where the cosine is negative.

Thus, cos(-512°) < 0

SB

CEMP049-22



Lecture 49: Page 22

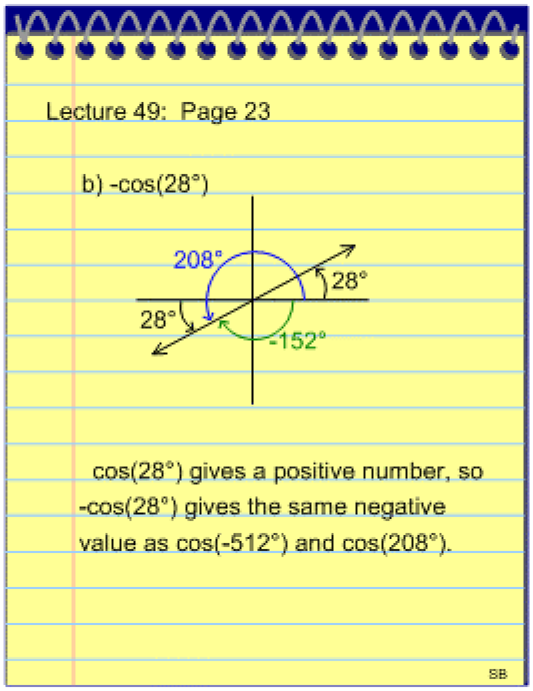
a) cos(208°)

208° = -152° ≡ These two angles are coterminal.

cos(208°) = cos(-512°)

SB

CEMP049-23



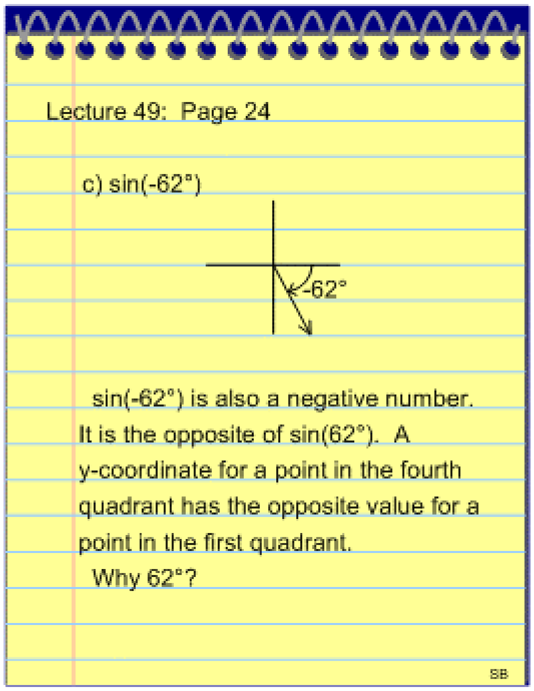
Lecture 49: Page 23

b) -cos(28°)

cos(28°) gives a positive number, so -cos(28°) gives the same negative value as cos(-512°) and cos(208°).

SB

CEMP049-24



Lecture 49: Page 24

c) sin(-62°)

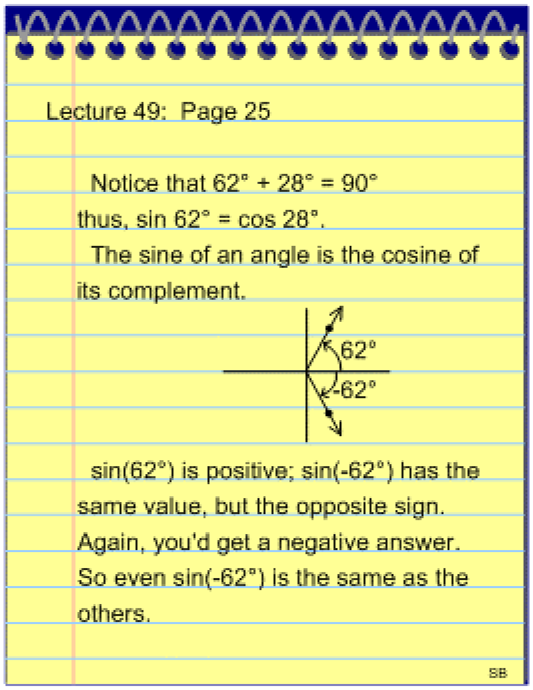
sin(-62°) is also a negative number. It is the opposite of sin(62°). A y-coordinate for a point in the fourth quadrant has the opposite value for a point in the first quadrant.

Why 62°?

SB

Lecture 49 Notes, Continued

CEMP049-25



Lecture 49: Page 25

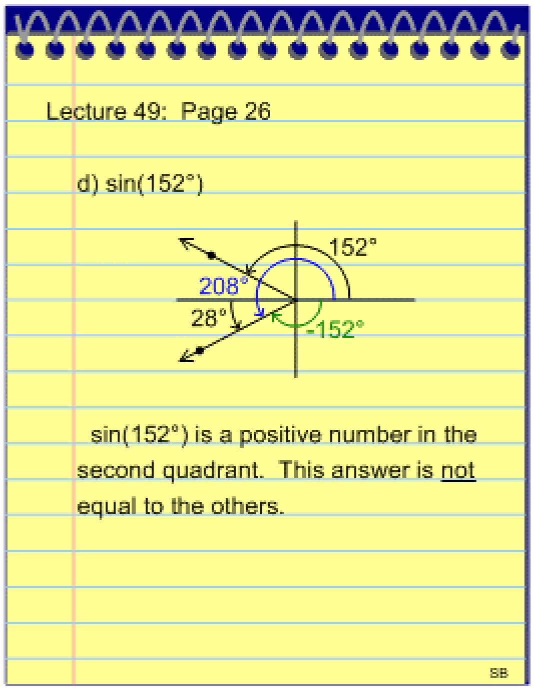
Notice that 62° + 28° = 90°
thus, sin 62° = cos 28°.

The sine of an angle is the cosine of its complement.

sin(62°) is positive; sin(-62°) has the same value, but the opposite sign. Again, you'd get a negative answer. So even sin(-62°) is the same as the others.

SB

CEMP049-26



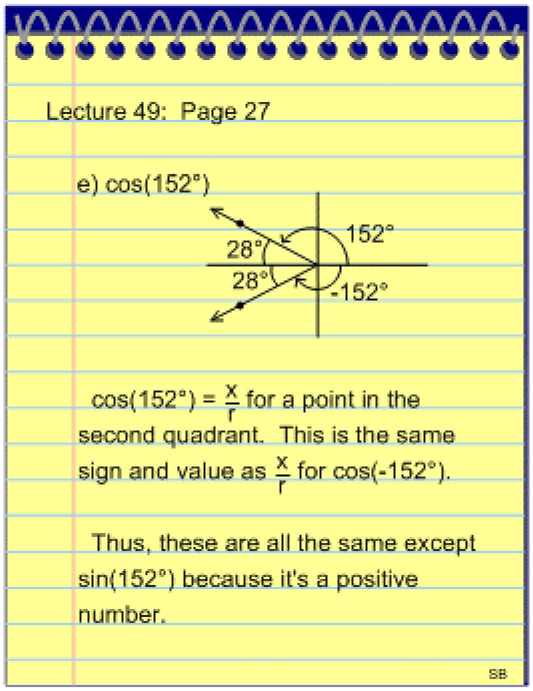
Lecture 49: Page 26

d) sin(152°)

sin(152°) is a positive number in the second quadrant. This answer is <u>not</u> equal to the others.

SB

CEMP049-27



Lecture 49: Page 27

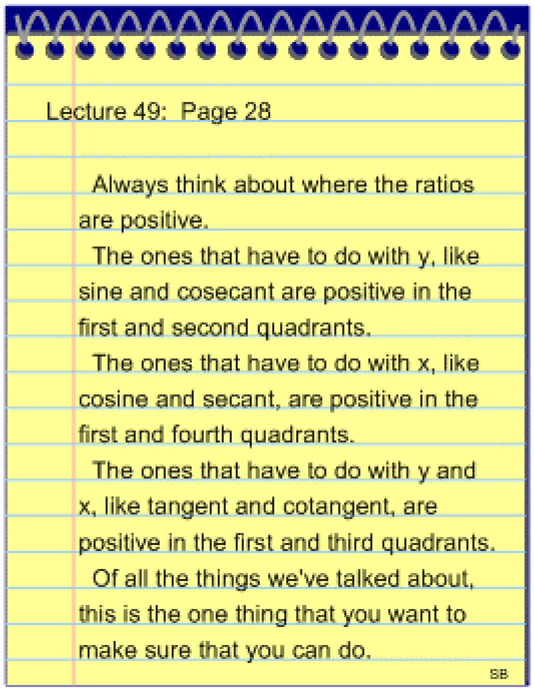
e) cos(152°)

cos(152°) = $\frac{x}{r}$ for a point in the second quadrant. This is the same sign and value as $\frac{x}{r}$ for cos(-152°).

Thus, these are all the same except sin(152°) because it's a positive number.

SB

CEMP049-28



Lecture 49: Page 28

Always think about where the ratios are positive.

The ones that have to do with y, like sine and cosecant are positive in the first and second quadrants.

The ones that have to do with x, like cosine and secant, are positive in the first and fourth quadrants.

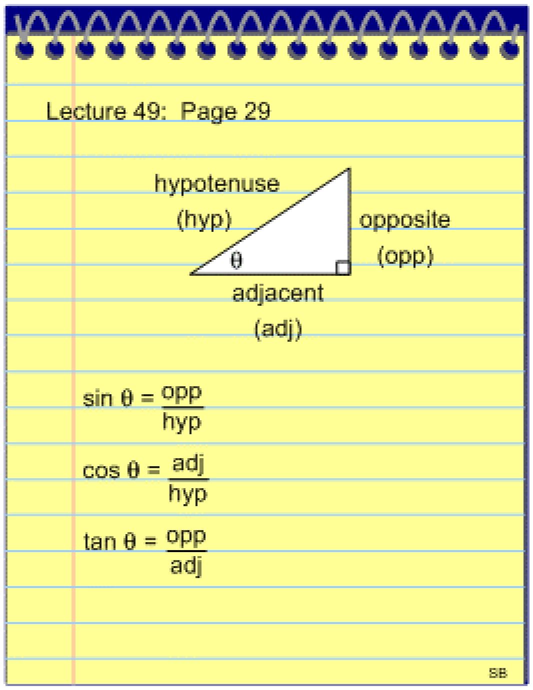
The ones that have to do with y and x, like tangent and cotangent, are positive in the first and third quadrants.

Of all the things we've talked about, this is the one thing that you want to make sure that you can do.

SB

Lecture 49 Notes, Continued

CEMP049-29



Lecture 49: Page 29

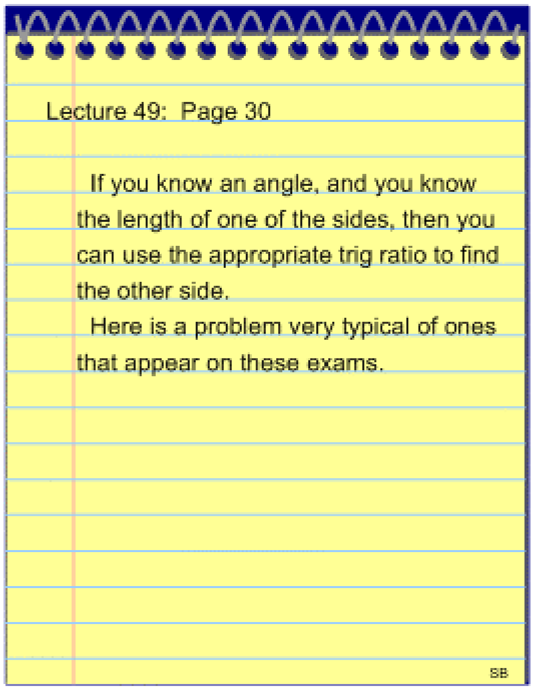
$\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

SB

CEMP049-30



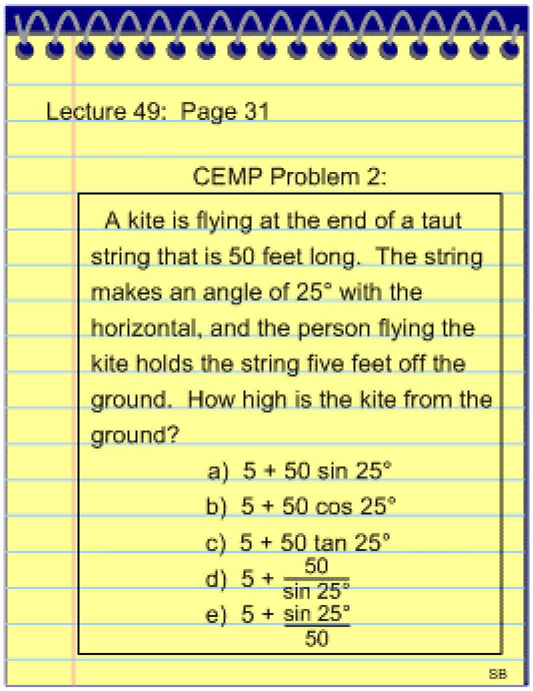
Lecture 49: Page 30

If you know an angle, and you know the length of one of the sides, then you can use the appropriate trig ratio to find the other side.

Here is a problem very typical of ones that appear on these exams.

SB

CEMP049-31



Lecture 49: Page 31

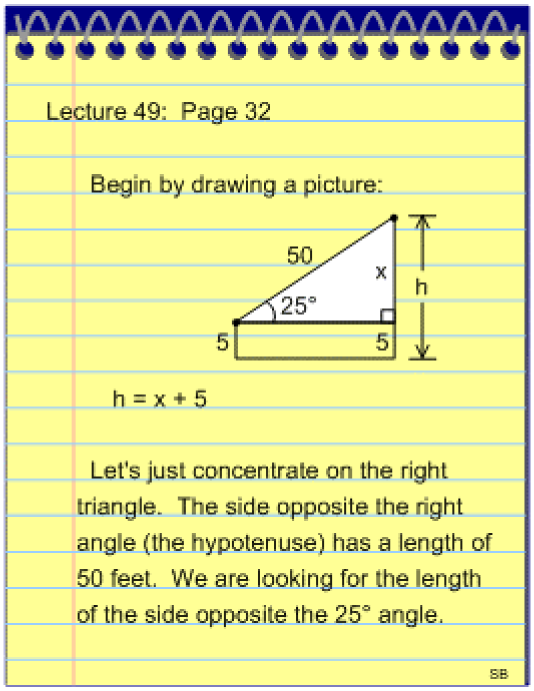
CEMP Problem 2:

A kite is flying at the end of a taut string that is 50 feet long. The string makes an angle of 25° with the horizontal, and the person flying the kite holds the string five feet off the ground. How high is the kite from the ground?

a) $5 + 50 \sin 25°$
b) $5 + 50 \cos 25°$
c) $5 + 50 \tan 25°$
d) $5 + \frac{50}{\sin 25°}$
e) $5 + \frac{\sin 25°}{50}$

SB

CEMP049-32



Lecture 49: Page 32

Begin by drawing a picture:

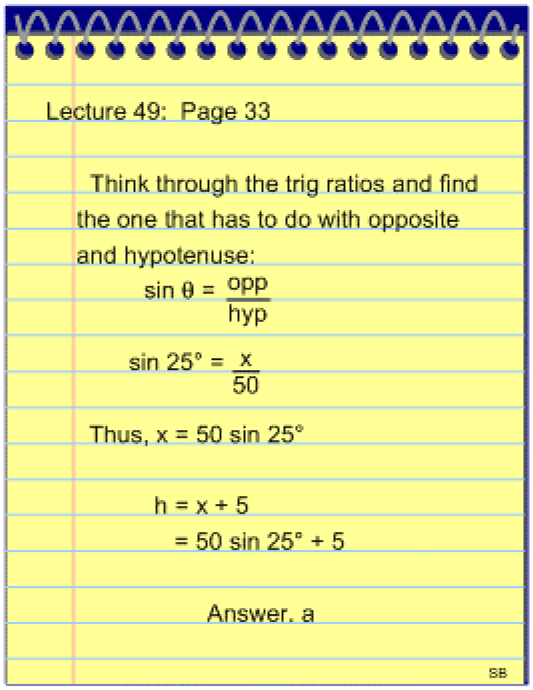
$h = x + 5$

Let's just concentrate on the right triangle. The side opposite the right angle (the hypotenuse) has a length of 50 feet. We are looking for the length of the side opposite the 25° angle.

SB

Lecture 49 Notes, Continued

CEMP049-33



Lecture 49: Page 33

Think through the trig ratios and find the one that has to do with opposite and hypotenuse:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 25° = \frac{x}{50}$$

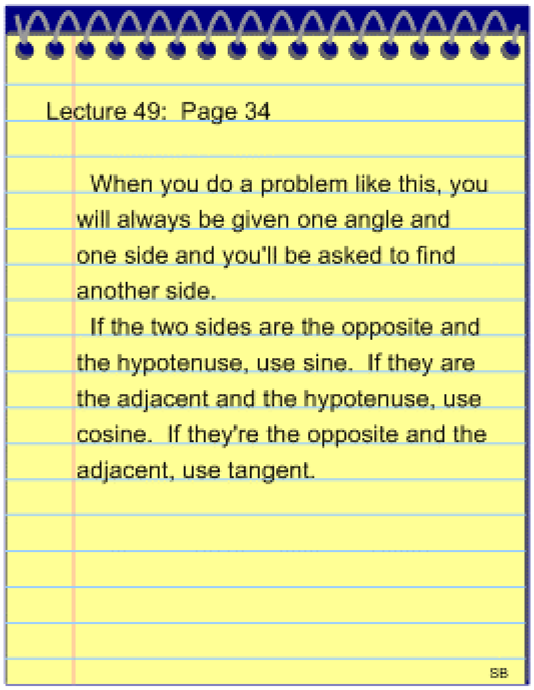
Thus, $x = 50 \sin 25°$

$$h = x + 5$$
$$= 50 \sin 25° + 5$$

Answer. a

SB

CEMP049-34



Lecture 49: Page 34

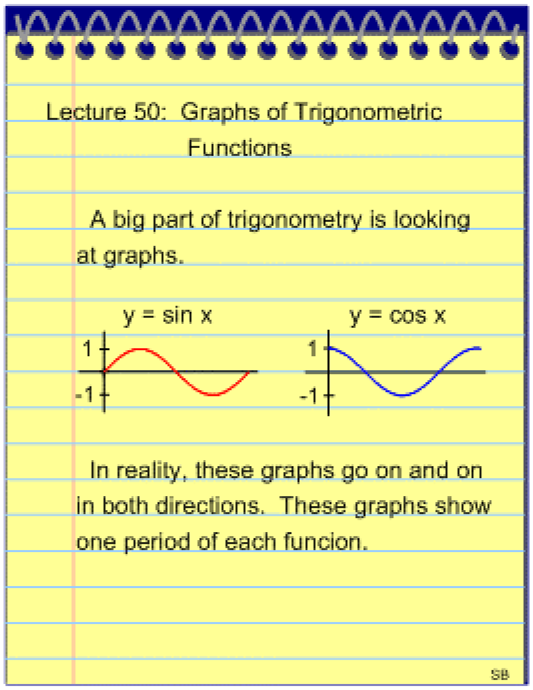
When you do a problem like this, you will always be given one angle and one side and you'll be asked to find another side.

If the two sides are the opposite and the hypotenuse, use sine. If they are the adjacent and the hypotenuse, use cosine. If they're the opposite and the adjacent, use tangent.

SB

Lecture 50 Notes

CEMP050-01



Lecture 50: Graphs of Trigonometric Functions

A big part of trigonometry is looking at graphs.

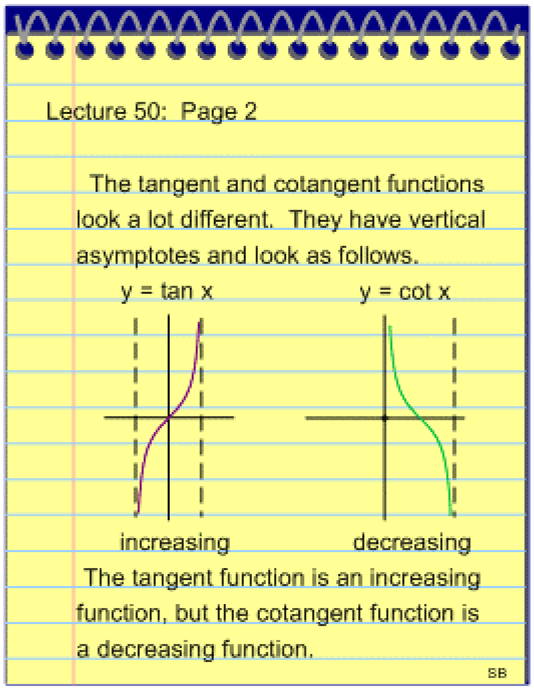
y = sin x

y = cos x

In reality, these graphs go on and on in both directions. These graphs show one period of each funcion.

SB

CEMP050-02



Lecture 50: Page 2

The tangent and cotangent functions look a lot different. They have vertical asymptotes and look as follows.

y = tan x

y = cot x

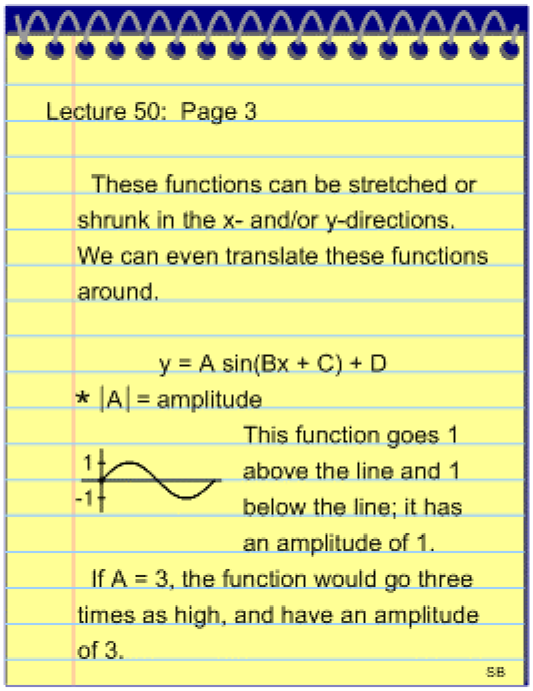
increasing

decreasing

The tangent function is an increasing function, but the cotangent function is a decreasing function.

SB

CEMP050-03



Lecture 50: Page 3

These functions can be stretched or shrunk in the x- and/or y-directions. We can even translate these functions around.

y = A sin(Bx + C) + D

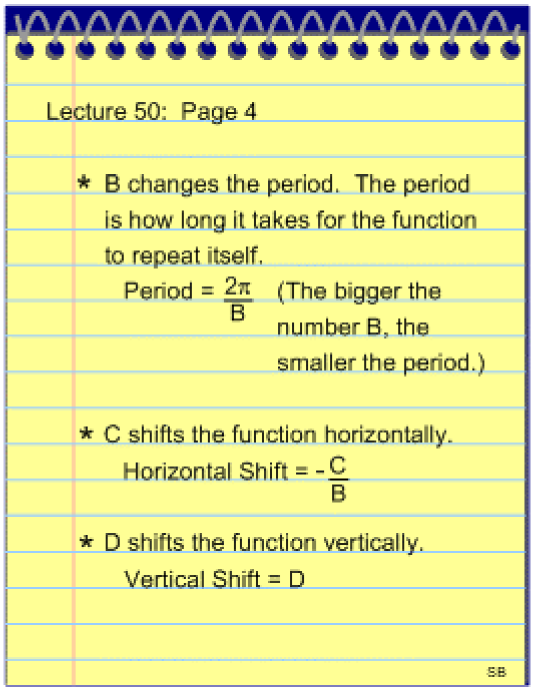
* $|A|$ = amplitude

This function goes 1 above the line and 1 below the line; it has an amplitude of 1.

If A = 3, the function would go three times as high, and have an amplitude of 3.

SB

CEMP050-04



Lecture 50: Page 4

* B changes the period. The period is how long it takes for the function to repeat itself.

Period = $\frac{2\pi}{B}$ (The bigger the number B, the smaller the period.)

* C shifts the function horizontally.

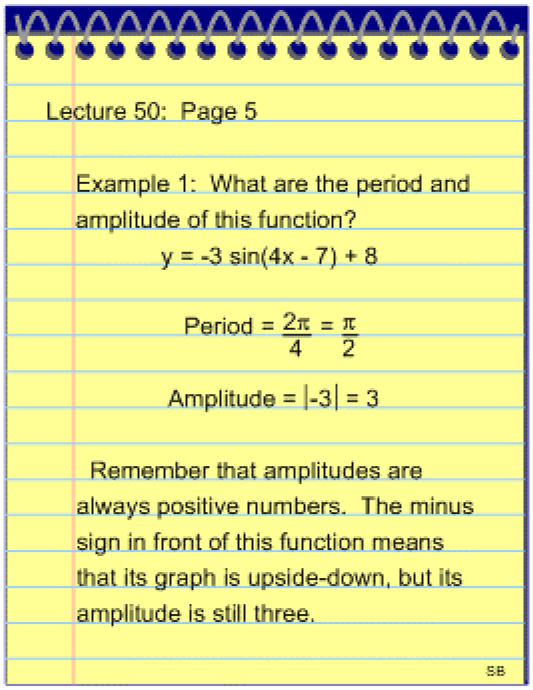
Horizontal Shift = $-\frac{C}{B}$

* D shifts the function vertically.

Vertical Shift = D

SB

CEMP050-05



Lecture 50: Page 5

Example 1: What are the period and amplitude of this function?

$y = -3\sin(4x - 7) + 8$

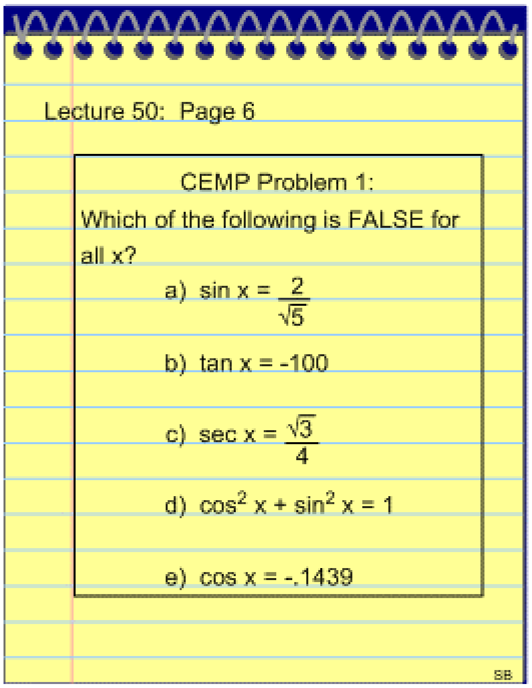
Period = $\frac{2\pi}{4} = \frac{\pi}{2}$

Amplitude = $|-3| = 3$

Remember that amplitudes are always positive numbers. The minus sign in front of this function means that its graph is upside-down, but its amplitude is still three.

SB

CEMP050-06



Lecture 50: Page 6

CEMP Problem 1:

Which of the following is FALSE for all x?

a) $\sin x = \frac{2}{\sqrt{5}}$

b) $\tan x = -100$

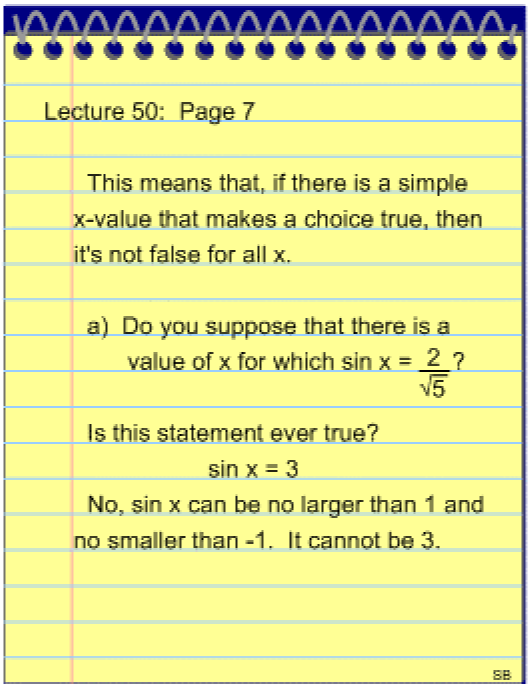
c) $\sec x = \frac{\sqrt{3}}{4}$

d) $\cos^2 x + \sin^2 x = 1$

e) $\cos x = -.1439$

SB

CEMP050-07



Lecture 50: Page 7

This means that, if there is a simple x-value that makes a choice true, then it's not false for all x.

a) Do you suppose that there is a value of x for which $\sin x = \frac{2}{\sqrt{5}}$?

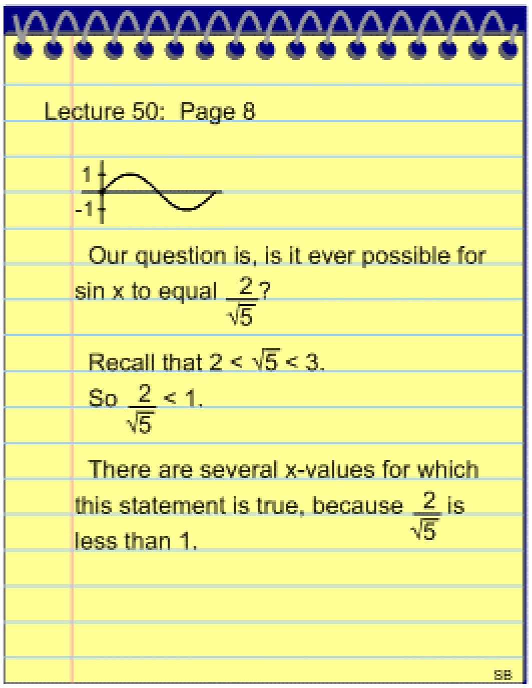
Is this statement ever true?

$\sin x = 3$

No, sin x can be no larger than 1 and no smaller than -1. It cannot be 3.

SB

CEMP050-08



Lecture 50: Page 8

Our question is, is it ever possible for sin x to equal $\frac{2}{\sqrt{5}}$?

Recall that $2 < \sqrt{5} < 3$.

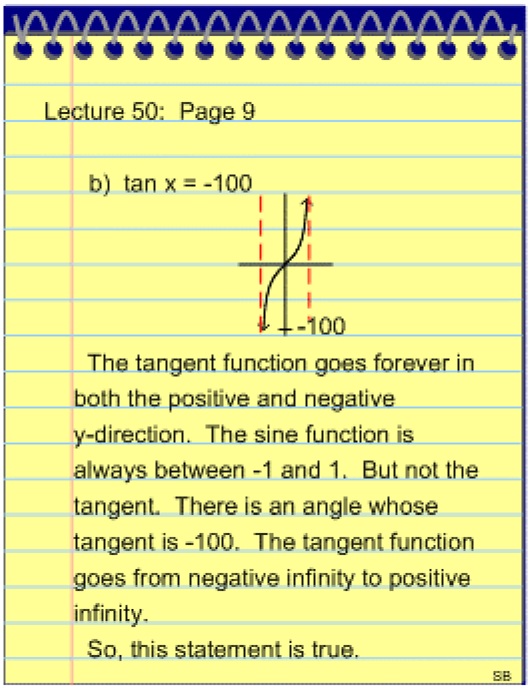
So $\frac{2}{\sqrt{5}} < 1$.

There are several x-values for which this statement is true, because $\frac{2}{\sqrt{5}}$ is less than 1.

SB

Lecture 50 Notes, Continued

CEMP050-09



Lecture 50: Page 9

b) $\tan x = -100$

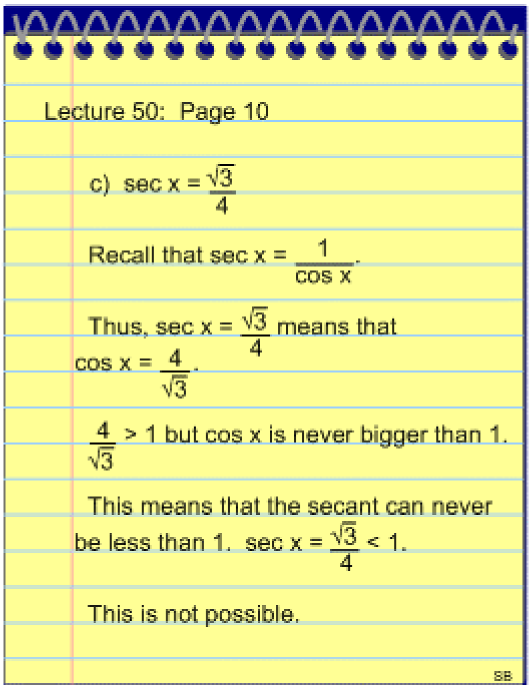
-100

The tangent function goes forever in both the positive and negative y-direction. The sine function is always between -1 and 1. But not the tangent. There is an angle whose tangent is -100. The tangent function goes from negative infinity to positive infinity.

So, this statement is true.

SB

CEMP050-10



Lecture 50: Page 10

c) $\sec x = \frac{\sqrt{3}}{4}$

Recall that $\sec x = \frac{1}{\cos x}$.

Thus, $\sec x = \frac{\sqrt{3}}{4}$ means that $\cos x = \frac{4}{\sqrt{3}}$.

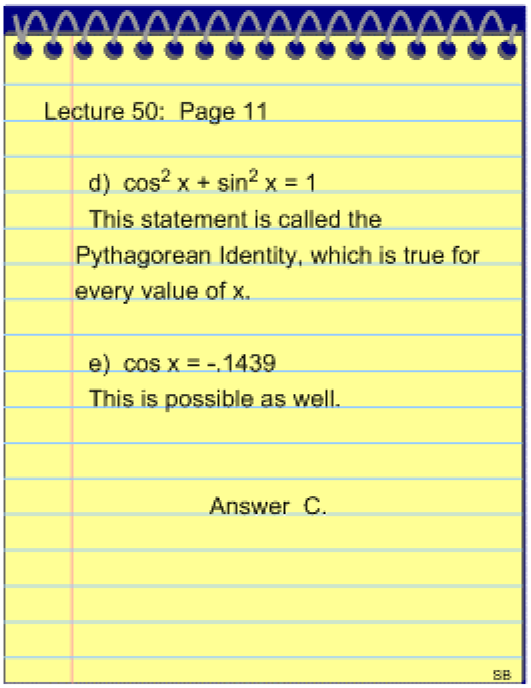
$\frac{4}{\sqrt{3}} > 1$ but cos x is never bigger than 1.

This means that the secant can never be less than 1. $\sec x = \frac{\sqrt{3}}{4} < 1$.

This is not possible.

SB

CEMP050-11



Lecture 50: Page 11

d) $\cos^2 x + \sin^2 x = 1$

This statement is called the Pythagorean Identity, which is true for every value of x.

e) $\cos x = -.1439$

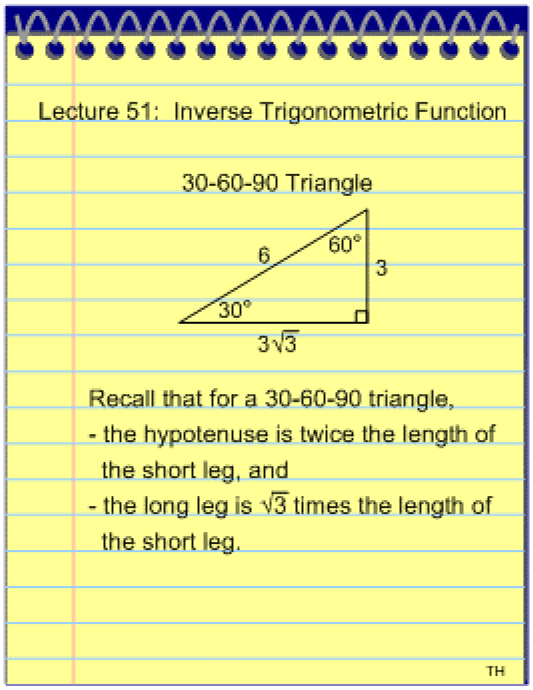
This is possible as well.

Answer C.

SB

Lecture 51 Notes

CEMP051-01

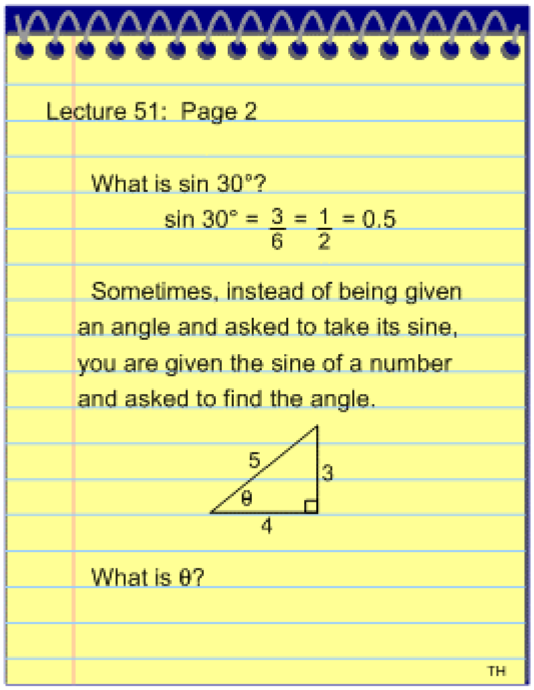


Lecture 51: Inverse Trigonometric Function

30-60-90 Triangle

Recall that for a 30-60-90 triangle,
- the hypotenuse is twice the length of the short leg, and
- the long leg is $\sqrt{3}$ times the length of the short leg.

CEMP051-02



Lecture 51: Page 2

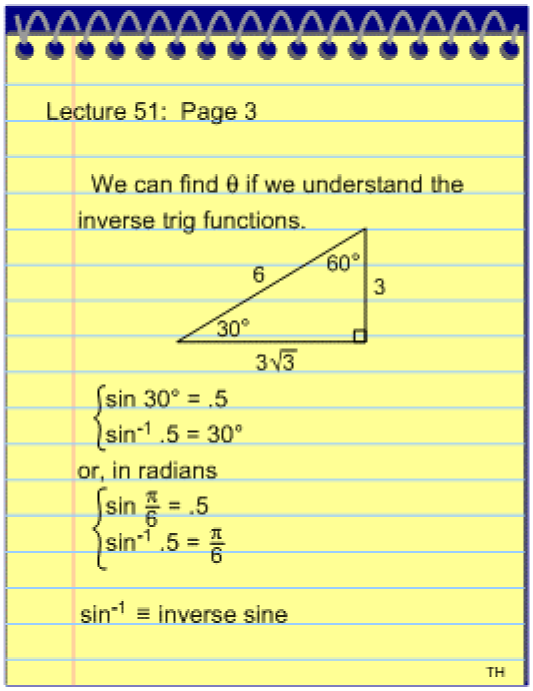
What is sin 30°?

$$\sin 30° = \frac{3}{6} = \frac{1}{2} = 0.5$$

Sometimes, instead of being given an angle and asked to take its sine, you are given the sine of a number and asked to find the angle.

What is θ?

CEMP051-03



Lecture 51: Page 3

We can find θ if we understand the inverse trig functions.

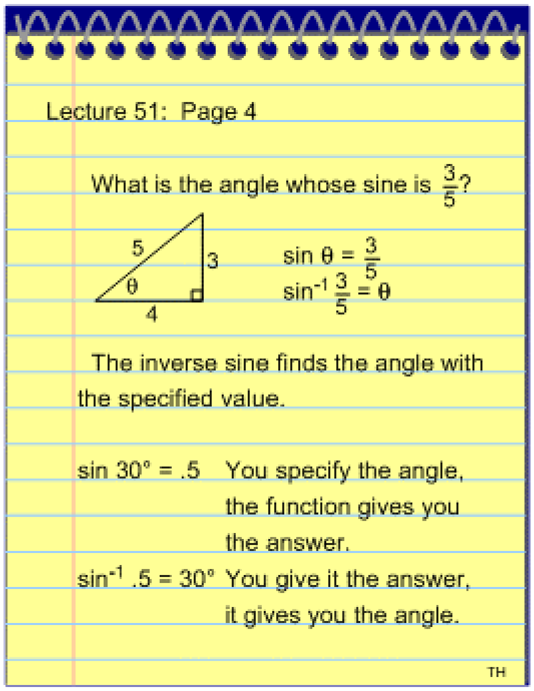
$$\begin{cases} \sin 30° = .5 \\ \sin^{-1} .5 = 30° \end{cases}$$

or, in radians

$$\begin{cases} \sin \frac{\pi}{6} = .5 \\ \sin^{-1} .5 = \frac{\pi}{6} \end{cases}$$

$\sin^{-1} \equiv$ inverse sine

CEMP051-04



Lecture 51: Page 4

What is the angle whose sine is $\frac{3}{5}$?

$$\sin \theta = \frac{3}{5}$$

$$\sin^{-1} \frac{3}{5} = \theta$$

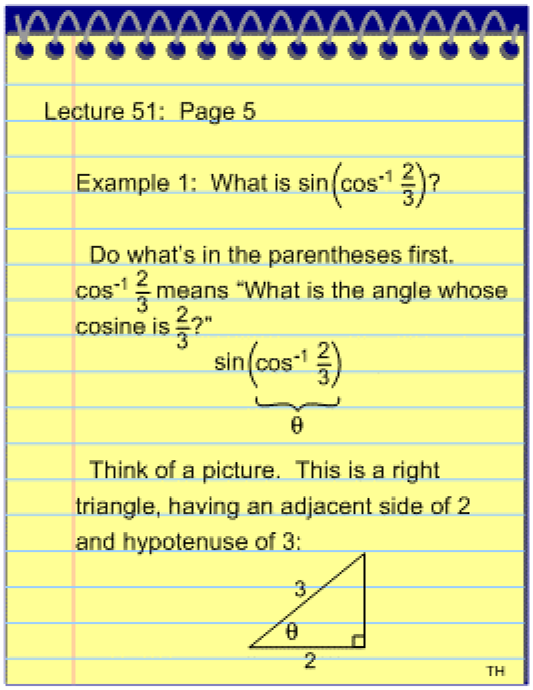
The inverse sine finds the angle with the specified value.

$\sin 30° = .5$ You specify the angle, the function gives you the answer.

$\sin^{-1} .5 = 30°$ You give it the answer, it gives you the angle.

Lecture 51 Notes, Continued

CEMP051-05



Lecture 51: Page 5

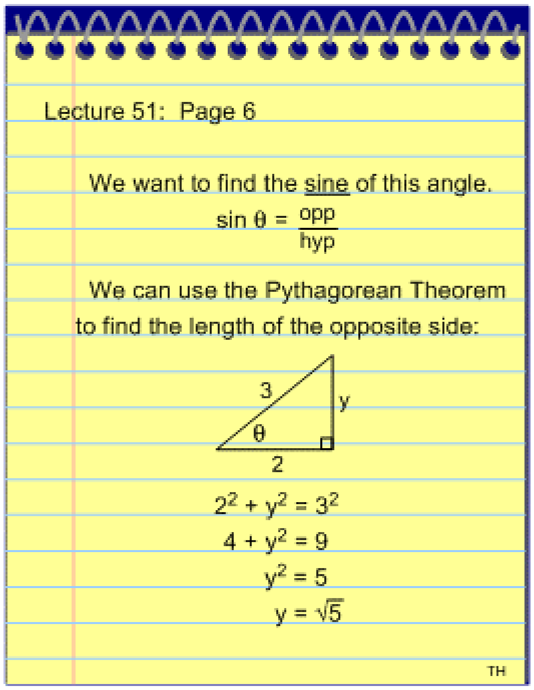
Example 1: What is $\sin\left(\cos^{-1}\frac{2}{3}\right)$?

Do what's in the parentheses first. $\cos^{-1}\frac{2}{3}$ means "What is the angle whose cosine is $\frac{2}{3}$?"

$$\sin\underbrace{\left(\cos^{-1}\frac{2}{3}\right)}_{\theta}$$

Think of a picture. This is a right triangle, having an adjacent side of 2 and hypotenuse of 3:

CEMP051-06



Lecture 51: Page 6

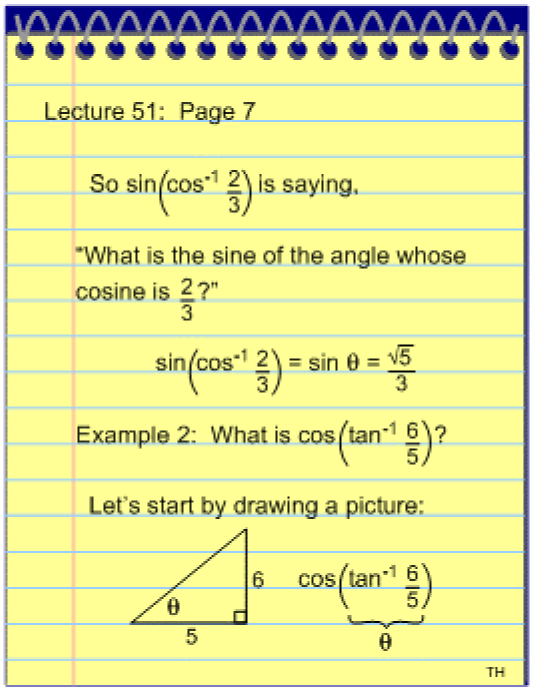
We want to find the sine of this angle.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

We can use the Pythagorean Theorem to find the length of the opposite side:

$$2^2 + y^2 = 3^2$$
$$4 + y^2 = 9$$
$$y^2 = 5$$
$$y = \sqrt{5}$$

CEMP051-07



Lecture 51: Page 7

So $\sin\left(\cos^{-1}\frac{2}{3}\right)$ is saying,

"What is the sine of the angle whose cosine is $\frac{2}{3}$?"

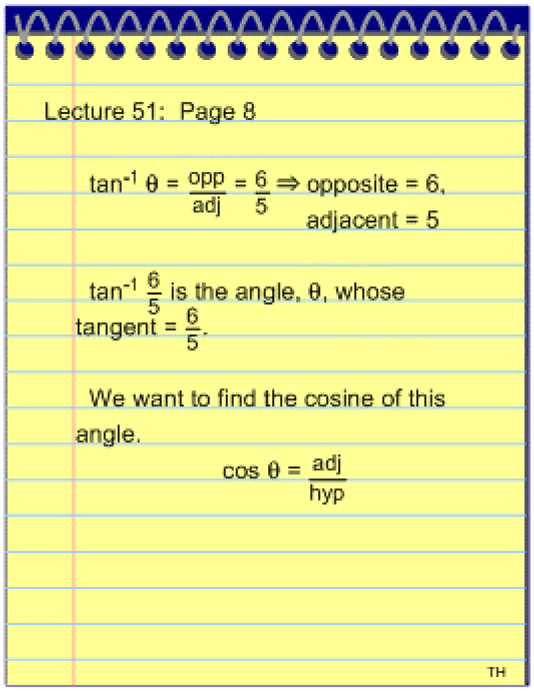
$$\sin\left(\cos^{-1}\frac{2}{3}\right) = \sin\theta = \frac{\sqrt{5}}{3}$$

Example 2: What is $\cos\left(\tan^{-1}\frac{6}{5}\right)$?

Let's start by drawing a picture:

$$\cos\underbrace{\left(\tan^{-1}\frac{6}{5}\right)}_{\theta}$$

CEMP051-08



Lecture 51: Page 8

$\tan^{-1}\theta = \frac{\text{opp}}{\text{adj}} = \frac{6}{5} \Rightarrow$ opposite = 6,
adjacent = 5

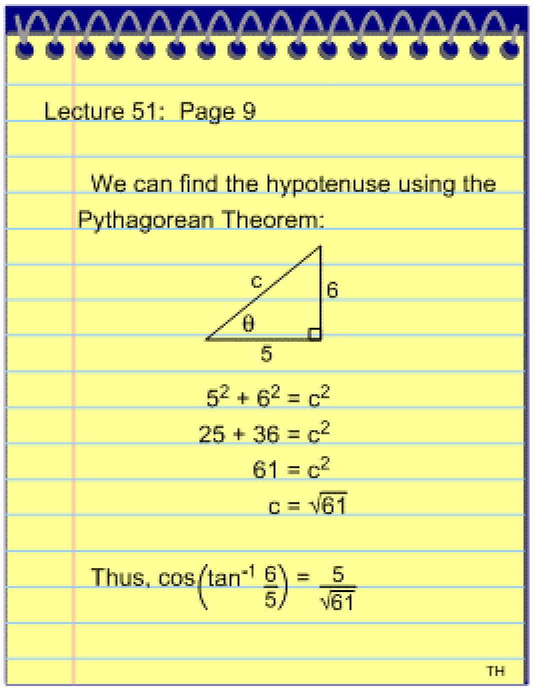
$\tan^{-1}\frac{6}{5}$ is the angle, θ, whose tangent = $\frac{6}{5}$.

We want to find the cosine of this angle.

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

Lecture 51 Notes, Continued

CEMP051-09



Lecture 51: Page 9

We can find the hypotenuse using the Pythagorean Theorem:

$$5^2 + 6^2 = c^2$$

$$25 + 36 = c^2$$

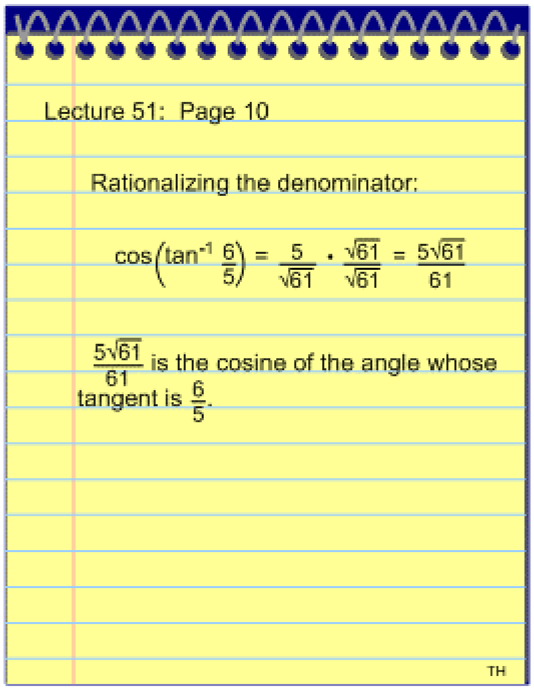
$$61 = c^2$$

$$c = \sqrt{61}$$

Thus, $\cos\left(\tan^{-1} \frac{6}{5}\right) = \frac{5}{\sqrt{61}}$

TH

CEMP051-10



Lecture 51: Page 10

Rationalizing the denominator:

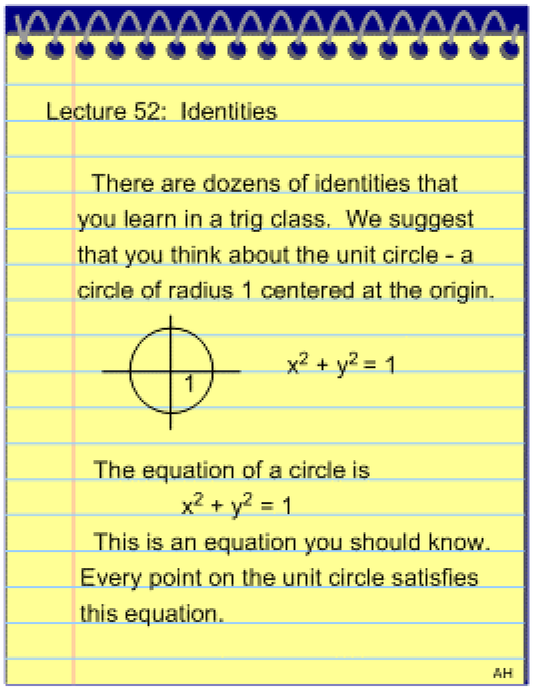
$$\cos\left(\tan^{-1} \frac{6}{5}\right) = \frac{5}{\sqrt{61}} \cdot \frac{\sqrt{61}}{\sqrt{61}} = \frac{5\sqrt{61}}{61}$$

$\frac{5\sqrt{61}}{61}$ is the cosine of the angle whose tangent is $\frac{6}{5}$.

TH

Lecture 52 Notes

CEMP052-01



Lecture 52: Identities

There are dozens of identities that you learn in a trig class. We suggest that you think about the unit circle - a circle of radius 1 centered at the origin.

$x^2 + y^2 = 1$

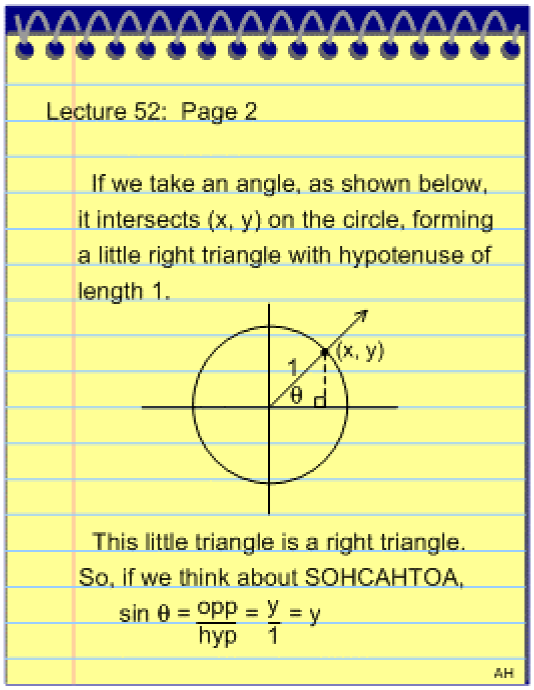
The equation of a circle is

$$x^2 + y^2 = 1$$

This is an equation you should know. Every point on the unit circle satisfies this equation.

AH

CEMP052-02



Lecture 52: Page 2

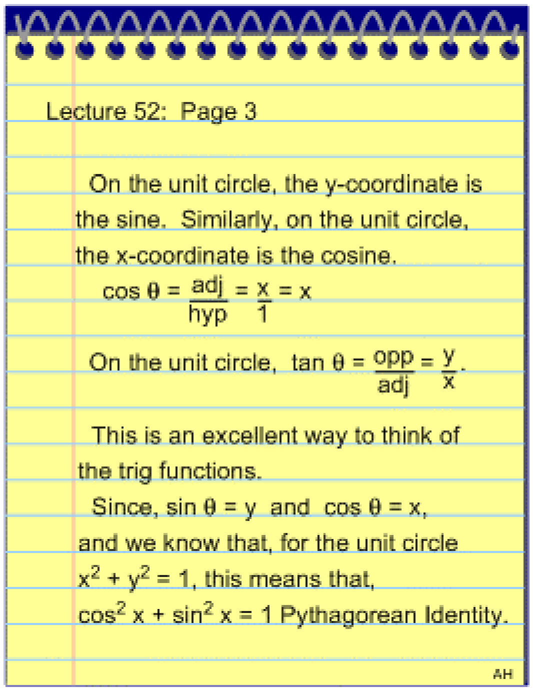
If we take an angle, as shown below, it intersects (x, y) on the circle, forming a little right triangle with hypotenuse of length 1.

This little triangle is a right triangle. So, if we think about SOHCAHTOA,

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

AH

CEMP052-03



Lecture 52: Page 3

On the unit circle, the y-coordinate is the sine. Similarly, on the unit circle, the x-coordinate is the cosine.

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

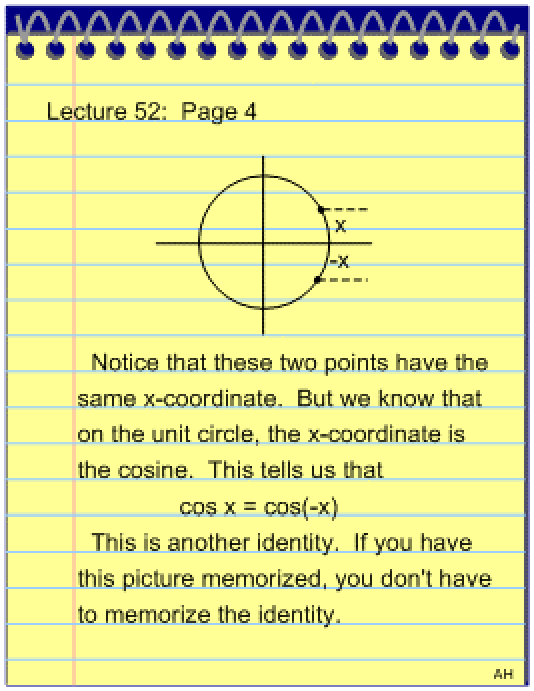
On the unit circle, $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$.

This is an excellent way to think of the trig functions.

Since, $\sin \theta = y$ and $\cos \theta = x$, and we know that, for the unit circle $x^2 + y^2 = 1$, this means that, $\cos^2 x + \sin^2 x = 1$ Pythagorean Identity.

AH

CEMP052-04



Lecture 52: Page 4

Notice that these two points have the same x-coordinate. But we know that on the unit circle, the x-coordinate is the cosine. This tells us that

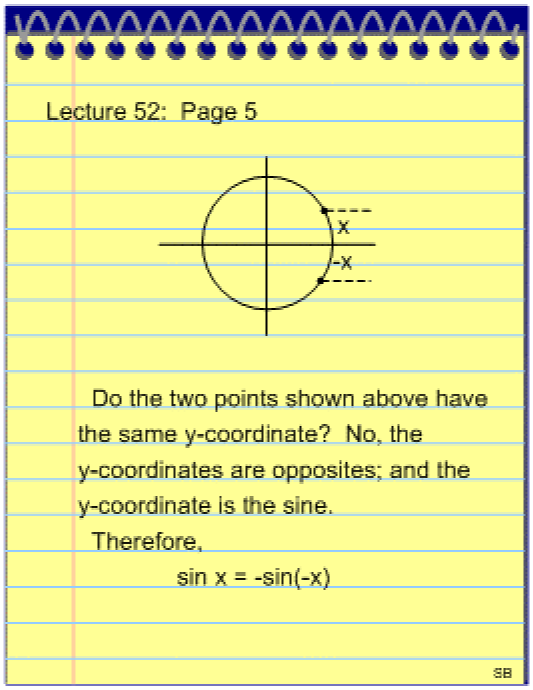
$$\cos x = \cos(-x)$$

This is another identity. If you have this picture memorized, you don't have to memorize the identity.

AH

Lecture 52 Notes, Continued

CEMP052-05



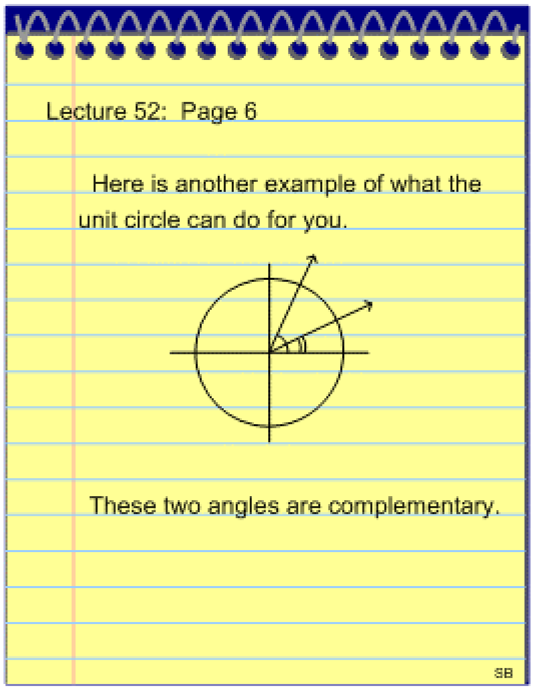
Lecture 52: Page 5

Do the two points shown above have the same y-coordinate? No, the y-coordinates are opposites; and the y-coordinate is the sine.

Therefore,

$$\sin x = -\sin(-x)$$

CEMP052-06

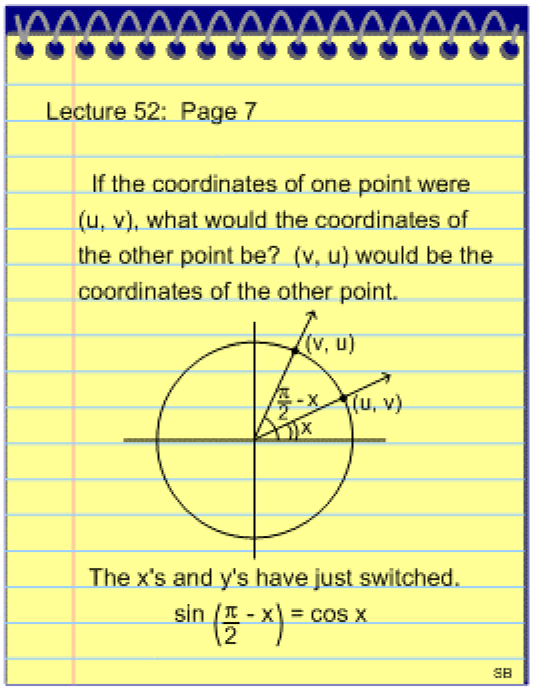


Lecture 52: Page 6

Here is another example of what the unit circle can do for you.

These two angles are complementary.

CEMP052-07



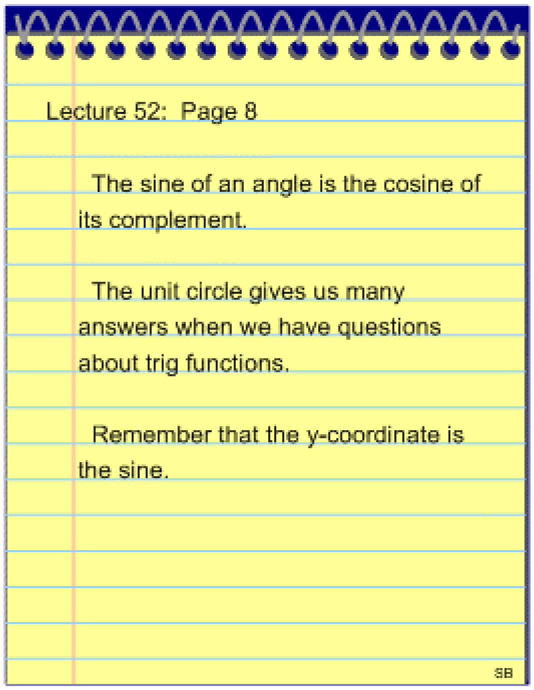
Lecture 52: Page 7

If the coordinates of one point were (u, v), what would the coordinates of the other point be? (v, u) would be the coordinates of the other point.

The x's and y's have just switched.

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

CEMP052-08



Lecture 52: Page 8

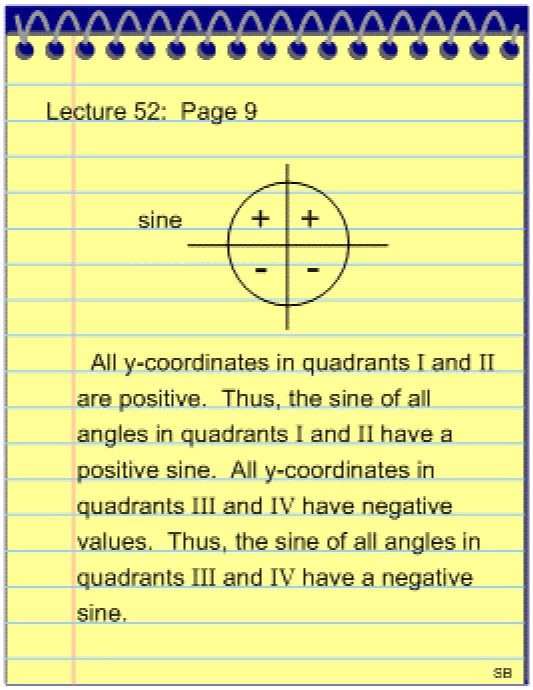
The sine of an angle is the cosine of its complement.

The unit circle gives us many answers when we have questions about trig functions.

Remember that the y-coordinate is the sine.

Lecture 52 Notes, Continued

CEMP052-09

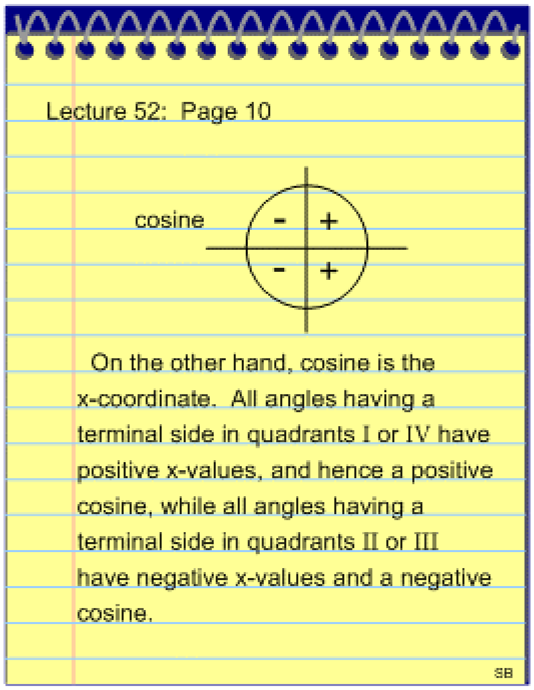


Lecture 52: Page 9

All y-coordinates in quadrants I and II are positive. Thus, the sine of all angles in quadrants I and II have a positive sine. All y-coordinates in quadrants III and IV have negative values. Thus, the sine of all angles in quadrants III and IV have a negative sine.

SB

CEMP052-10

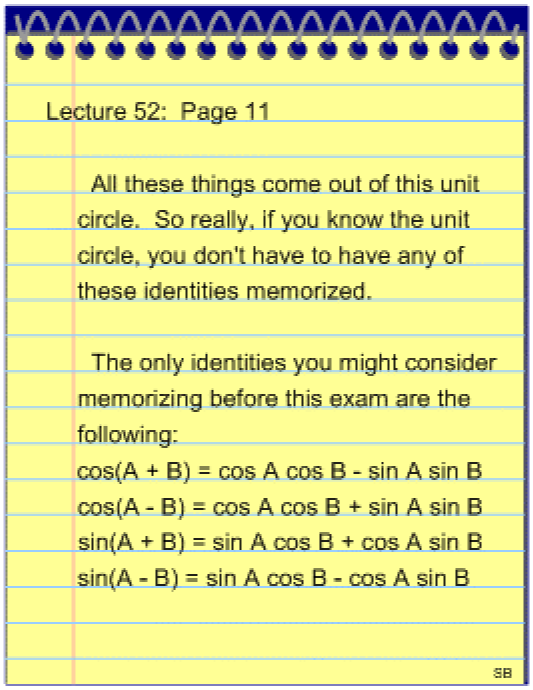


Lecture 52: Page 10

On the other hand, cosine is the x-coordinate. All angles having a terminal side in quadrants I or IV have positive x-values, and hence a positive cosine, while all angles having a terminal side in quadrants II or III have negative x-values and a negative cosine.

SB

CEMP052-11



Lecture 52: Page 11

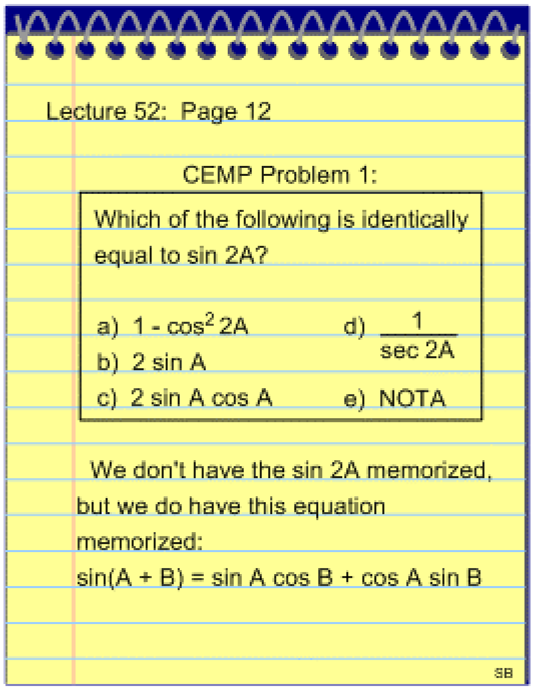
All these things come out of this unit circle. So really, if you know the unit circle, you don't have to have any of these identities memorized.

The only identities you might consider memorizing before this exam are the following:

cos(A + B) = cos A cos B - sin A sin B
cos(A - B) = cos A cos B + sin A sin B
sin(A + B) = sin A cos B + cos A sin B
sin(A - B) = sin A cos B - cos A sin B

SB

CEMP052-12



Lecture 52: Page 12

CEMP Problem 1:

Which of the following is identically equal to sin 2A?

a) $1 - \cos^2 2A$
b) 2 sin A
c) 2 sin A cos A
d) $\frac{1}{\sec 2A}$
e) NOTA

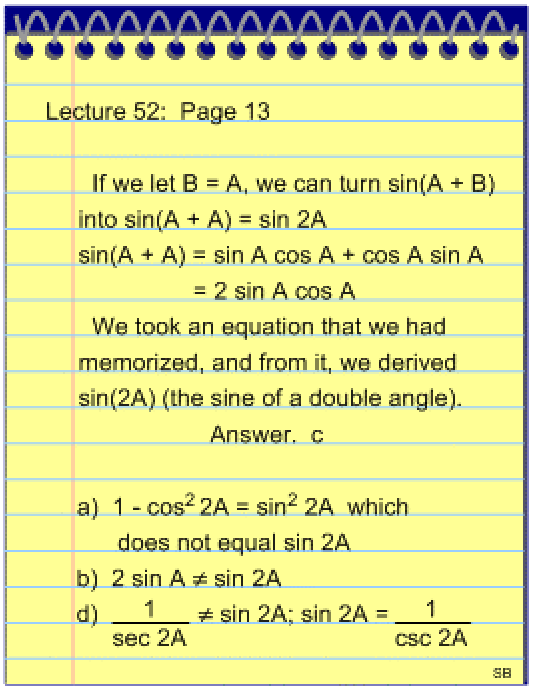
We don't have the sin 2A memorized, but we do have this equation memorized:

sin(A + B) = sin A cos B + cos A sin B

SB

Lecture 52 Notes, Continued

CEMP052-13



Lecture 52: Page 13

If we let $B = A$, we can turn $\sin(A + B)$ into $\sin(A + A) = \sin 2A$

$\sin(A + A) = \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

We took an equation that we had memorized, and from it, we derived $\sin(2A)$ (the sine of a double angle).

Answer. c

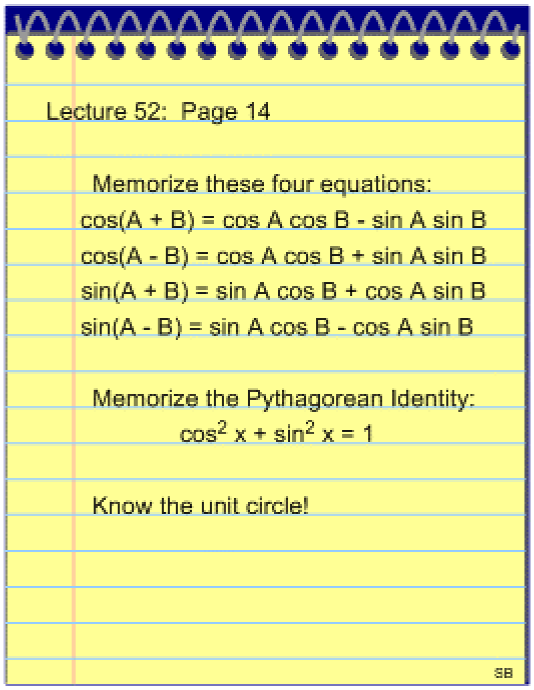
a) $1 - \cos^2 2A = \sin^2 2A$ which does not equal $\sin 2A$

b) $2 \sin A \neq \sin 2A$

d) $\frac{1}{\sec 2A} \neq \sin 2A$; $\sin 2A = \frac{1}{\csc 2A}$

SB

CEMP052-14



Lecture 52: Page 14

Memorize these four equations:

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

Memorize the Pythagorean Identity:

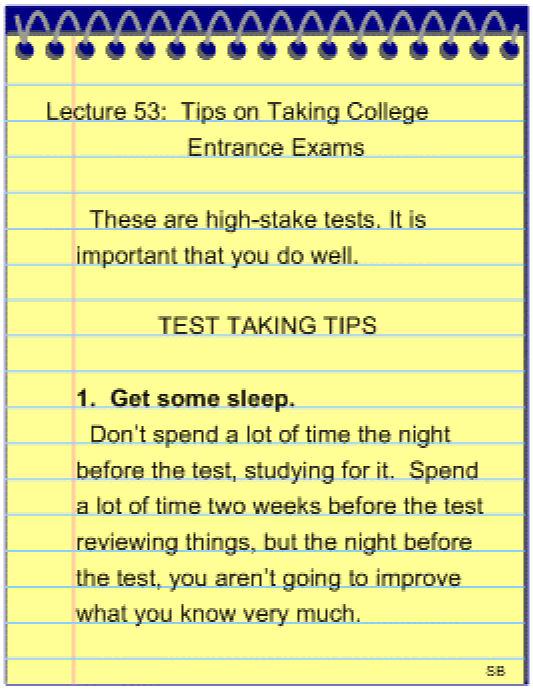
$\cos^2 x + \sin^2 x = 1$

Know the unit circle!

SB

Lecture 53 Notes

CEMP053-01



Lecture 53: Tips on Taking College Entrance Exams

These are high-stake tests. It is important that you do well.

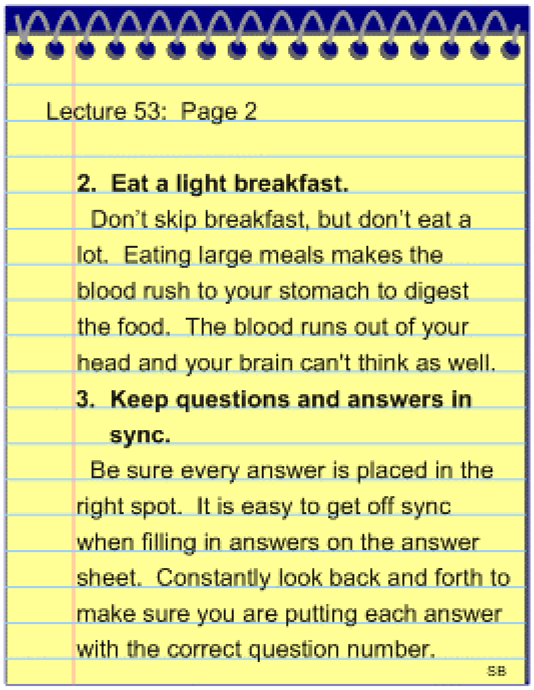
TEST TAKING TIPS

1. Get some sleep.

Don't spend a lot of time the night before the test, studying for it. Spend a lot of time two weeks before the test reviewing things, but the night before the test, you aren't going to improve what you know very much.

SB

CEMP053-02



Lecture 53: Page 2

2. Eat a light breakfast.

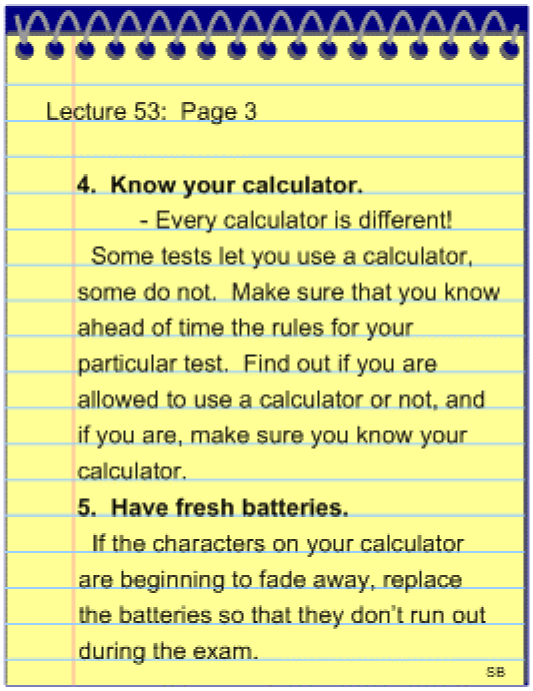
Don't skip breakfast, but don't eat a lot. Eating large meals makes the blood rush to your stomach to digest the food. The blood runs out of your head and your brain can't think as well.

3. Keep questions and answers in sync.

Be sure every answer is placed in the right spot. It is easy to get off sync when filling in answers on the answer sheet. Constantly look back and forth to make sure you are putting each answer with the correct question number.

SB

CEMP053-03



Lecture 53: Page 3

4. Know your calculator.

- Every calculator is different!

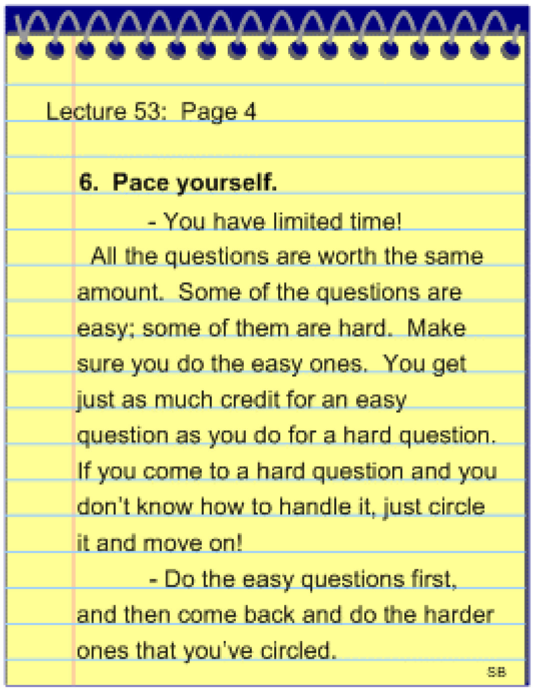
Some tests let you use a calculator, some do not. Make sure that you know ahead of time the rules for your particular test. Find out if you are allowed to use a calculator or not, and if you are, make sure you know your calculator.

5. Have fresh batteries.

If the characters on your calculator are beginning to fade away, replace the batteries so that they don't run out during the exam.

SB

CEMP053-04



Lecture 53: Page 4

6. Pace yourself.

- You have limited time!

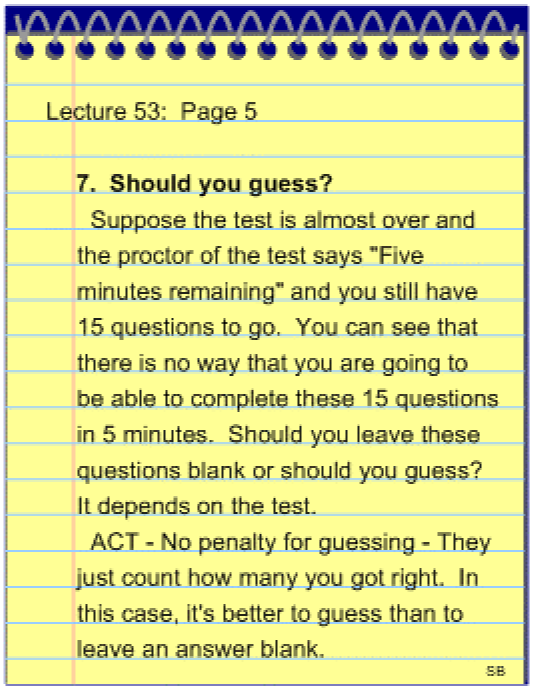
All the questions are worth the same amount. Some of the questions are easy; some of them are hard. Make sure you do the easy ones. You get just as much credit for an easy question as you do for a hard question. If you come to a hard question and you don't know how to handle it, just circle it and move on!

- Do the easy questions first, and then come back and do the harder ones that you've circled.

SB

Lecture 53 Notes, Continued

CEMP053-05



Lecture 53: Page 5

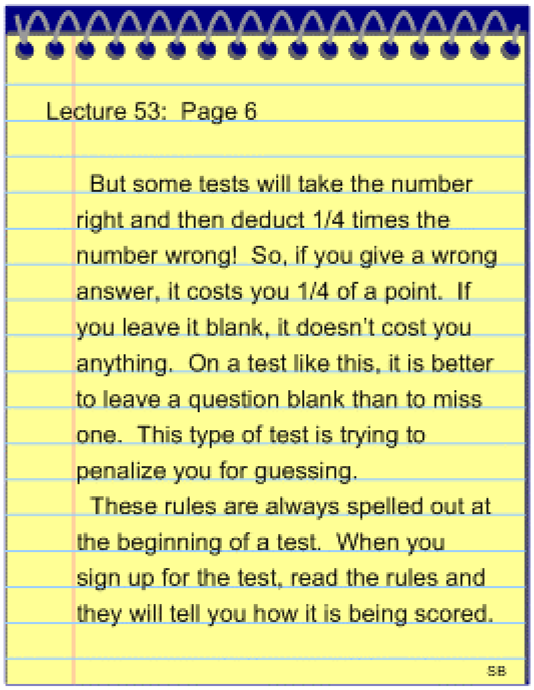
7. Should you guess?

Suppose the test is almost over and the proctor of the test says "Five minutes remaining" and you still have 15 questions to go. You can see that there is no way that you are going to be able to complete these 15 questions in 5 minutes. Should you leave these questions blank or should you guess? It depends on the test.

ACT - No penalty for guessing - They just count how many you got right. In this case, it's better to guess than to leave an answer blank.

SB

CEMP053-06



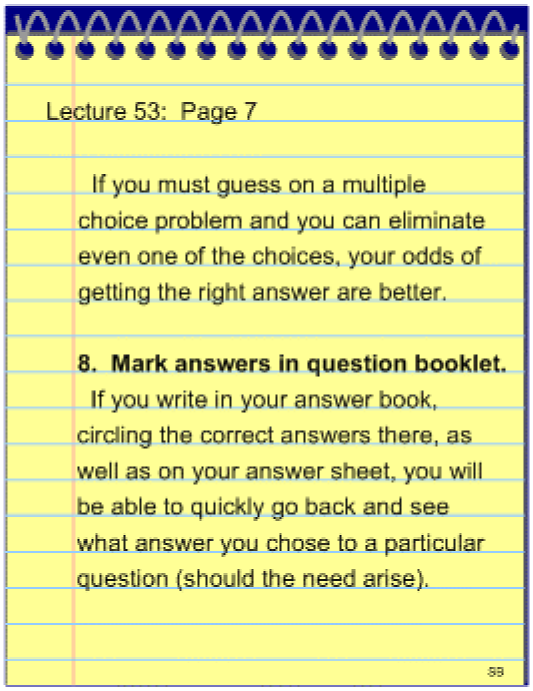
Lecture 53: Page 6

But some tests will take the number right and then deduct 1/4 times the number wrong! So, if you give a wrong answer, it costs you 1/4 of a point. If you leave it blank, it doesn't cost you anything. On a test like this, it is better to leave a question blank than to miss one. This type of test is trying to penalize you for guessing.

These rules are always spelled out at the beginning of a test. When you sign up for the test, read the rules and they will tell you how it is being scored.

SB

CEMP053-07



Lecture 53: Page 7

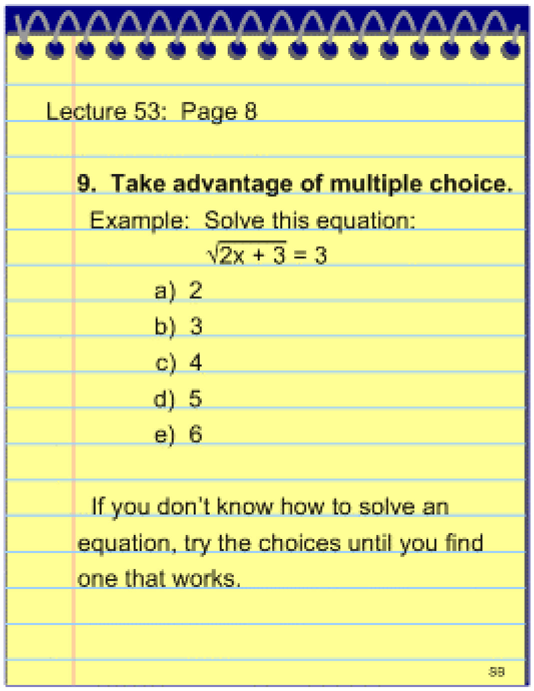
If you must guess on a multiple choice problem and you can eliminate even one of the choices, your odds of getting the right answer are better.

8. Mark answers in question booklet.

If you write in your answer book, circling the correct answers there, as well as on your answer sheet, you will be able to quickly go back and see what answer you chose to a particular question (should the need arise).

SB

CEMP053-08



Lecture 53: Page 8

9. Take advantage of multiple choice.

Example: Solve this equation:

$$\sqrt{2x + 3} = 3$$

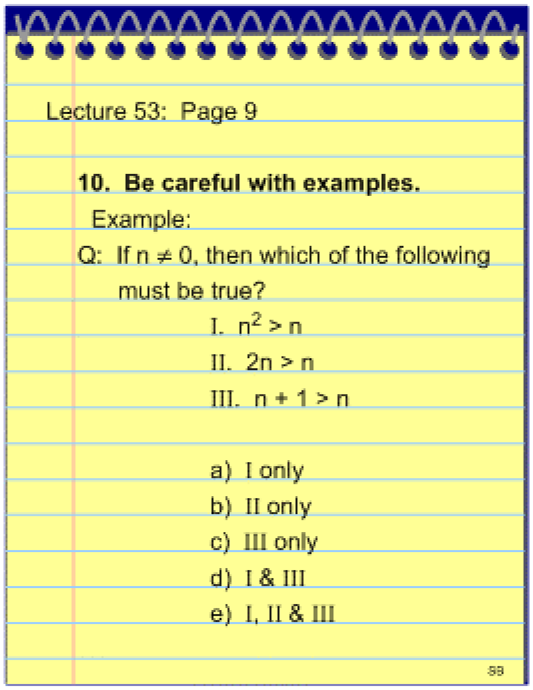
a) 2
b) 3
c) 4
d) 5
e) 6

If you don't know how to solve an equation, try the choices until you find one that works.

SB

Lecture 53 Notes, Continued

CEMP053-09



Lecture 53: Page 9

10. Be careful with examples.

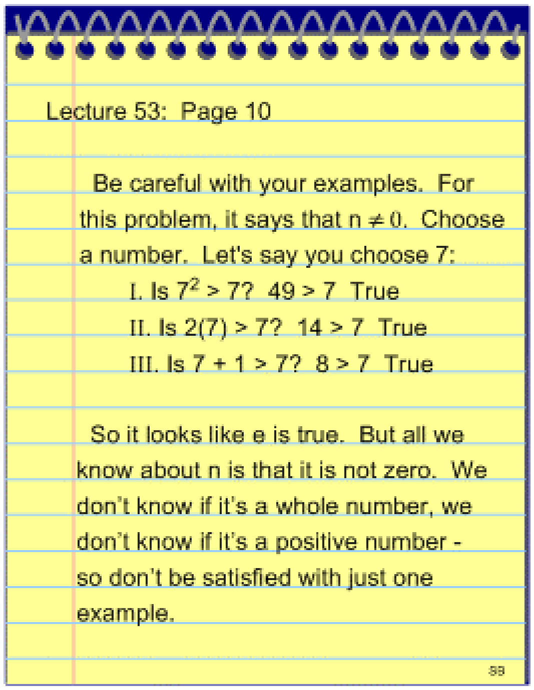
Example:

Q: If $n \neq 0$, then which of the following must be true?

I. $n^2 > n$
II. $2n > n$
III. $n + 1 > n$

a) I only
b) II only
c) III only
d) I & III
e) I, II & III

CEMP053-10



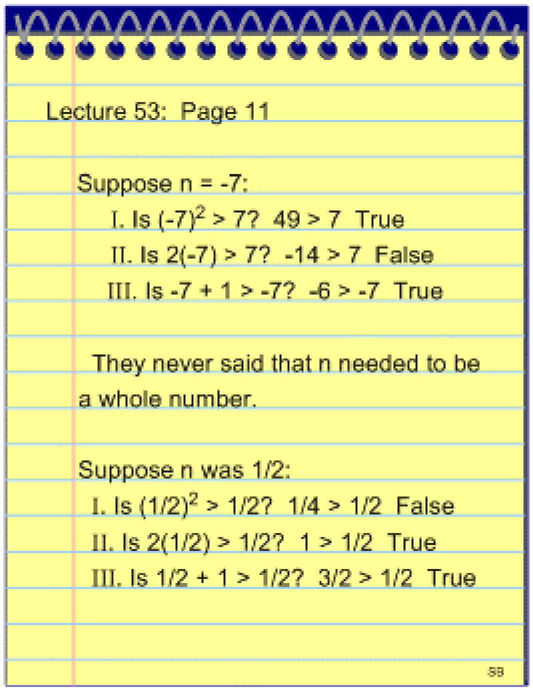
Lecture 53: Page 10

Be careful with your examples. For this problem, it says that $n \neq 0$. Choose a number. Let's say you choose 7:

I. Is $7^2 > 7$? $49 > 7$ True
II. Is $2(7) > 7$? $14 > 7$ True
III. Is $7 + 1 > 7$? $8 > 7$ True

So it looks like e is true. But all we know about n is that it is not zero. We don't know if it's a whole number, we don't know if it's a positive number - so don't be satisfied with just one example.

CEMP053-11



Lecture 53: Page 11

Suppose $n = -7$:

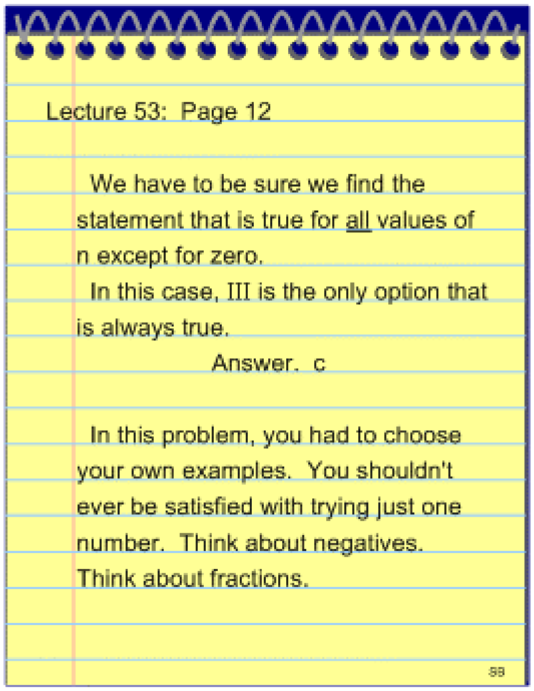
I. Is $(-7)^2 > 7$? $49 > 7$ True
II. Is $2(-7) > 7$? $-14 > 7$ False
III. Is $-7 + 1 > -7$? $-6 > -7$ True

They never said that n needed to be a whole number.

Suppose n was 1/2:

I. Is $(1/2)^2 > 1/2$? $1/4 > 1/2$ False
II. Is $2(1/2) > 1/2$? $1 > 1/2$ True
III. Is $1/2 + 1 > 1/2$? $3/2 > 1/2$ True

CEMP053-12



Lecture 53: Page 12

We have to be sure we find the statement that is true for all values of n except for zero.

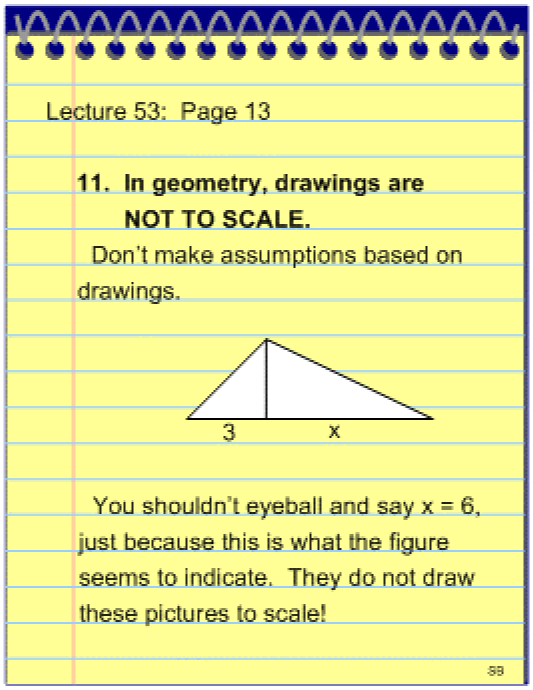
In this case, III is the only option that is always true.

Answer. c

In this problem, you had to choose your own examples. You shouldn't ever be satisfied with trying just one number. Think about negatives. Think about fractions.

Lecture 53 Notes, Continued

CEMP053-13



Lecture 53: Page 13

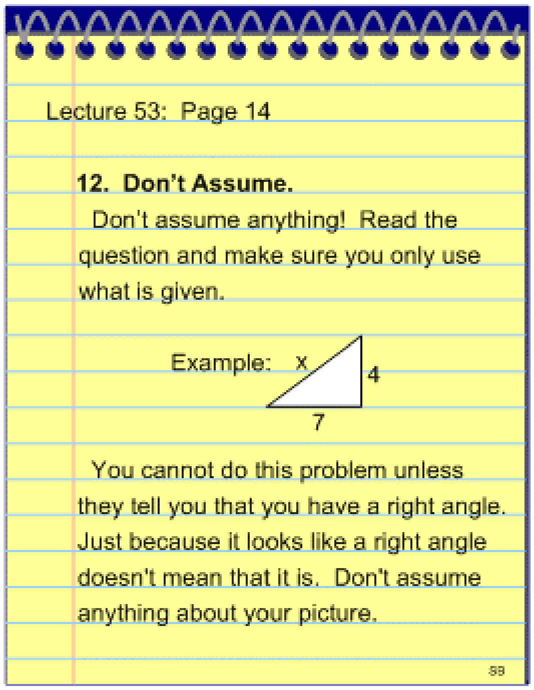
11. In geometry, drawings are NOT TO SCALE.

Don't make assumptions based on drawings.

3 x

You shouldn't eyeball and say x = 6, just because this is what the figure seems to indicate. They do not draw these pictures to scale!

CEMP053-14



Lecture 53: Page 14

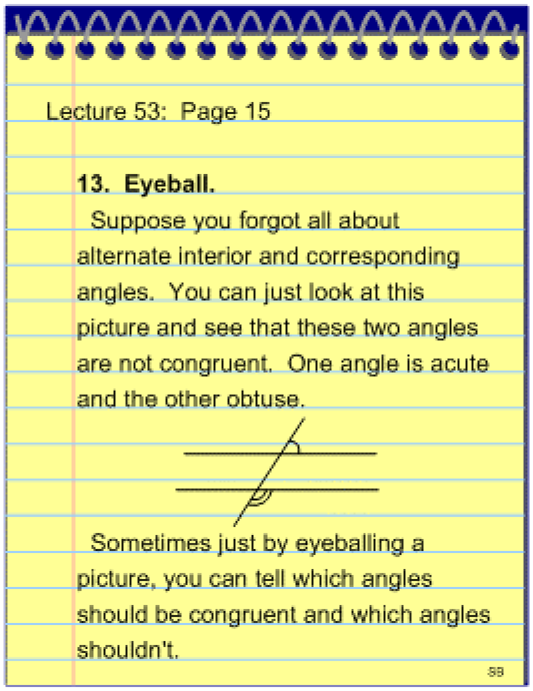
12. Don't Assume.

Don't assume anything! Read the question and make sure you only use what is given.

Example: x 4 7

You cannot do this problem unless they tell you that you have a right angle. Just because it looks like a right angle doesn't mean that it is. Don't assume anything about your picture.

CEMP053-15



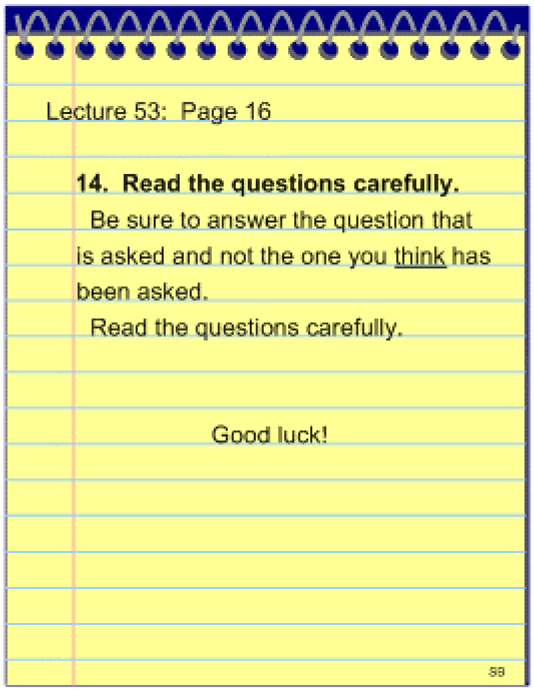
Lecture 53: Page 15

13. Eyeball.

Suppose you forgot all about alternate interior and corresponding angles. You can just look at this picture and see that these two angles are not congruent. One angle is acute and the other obtuse.

Sometimes just by eyeballing a picture, you can tell which angles should be congruent and which angles shouldn't.

CEMP053-16



Lecture 53: Page 16

14. Read the questions carefully.

Be sure to answer the question that is asked and not the one you think has been asked.

Read the questions carefully.

Good luck!